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Edited by Vladimir G. Turaev

Institut de Recherche Mathématique Avancée
Université Louis Pasteur et CNRS
7, rue René Descartes
67084 Strasbourg Cedex
France

Andrey Bolibrukh in 1980

Differential Equations and Quantum Groups

Andrey A. Bolibrukh Memorial Volume

Daniel Bertrand
Benjamin Enriquez
Claude Mitschi
Claude Sabbah
Reinhard Schäfke

Editors

European Mathematical Society

Editors:

Daniel Bertrand
Institut de Mathématiques
Université Pierre et Marie Curie
4, place Jussieu
75252 Paris Cedex 05
France

Claude Sabbah
CNRS, UMR 7640
Centre de Mathématiques Laurent Schwartz
École Polytechnique
91128 Palaiseau Cedex
France

Benjamin Enriquez
Claude Mitschi
Reinhard Schäfke
Institut de Recherche Mathématique Avancée
Université Louis Pasteur et CNRS
7, rue René Descartes
67084 Strasbourg Cedex
France

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European Mathematical Society Publishing House
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Preface

The present volume of IRMA Lectures is dedicated to the memory of Andrei Bolibrukh. It assembles original papers by authors who all have shared the privilege of personal, mathematical acquaintance with our late colleague. The subjects of the articles range from differential equations to quantum groups. They relate to questions such as the Riemann–Hilbert problem (Kostov), isomonodromic deformations (Dubrovin and Mazzocco), Painlevé equations (Boalch), integrability (Audin), summability (Balser), monodromy of KZ and CKV equations (Golubeva, Leksin), Bethe Ansatz and Schubert calculus (Belkale, Mukhin and Varchenko), differential Galois theory and differential algebraic groups (Cassidy and Singer, Umemura). This is exactly the palette of themes that Andrei Bolibrukh included in one of his last important achievements, namely the research cooperation program PICS (Programme International de Recherche Scientifique) between France and Russia, aimed at bringing together mathematicians from these fields. As this program started, Andrei Bolibrukh was already seriously ill, but even from his hospital room he kept thinking about new ideas and plans for this cooperation. He passed away on November 11, 2003 in Paris without having benefitted himself from this exchange program.

Andrei Andreevich Bolibrukh was born on January 30, 1950 in Moscow – one hundred years exactly after the birth of Sonia Kovalevskaia, whom he celebrated in January 2000 in a memorial conference. He received his mathematical education at the University of Moscow, with Postnikov and Chernavskii as thesis advisers. In 1989 Bolibrukh closed the chapter of a decade of doubts and uncertainty about Hilbert's 21st problem: he produced famous counterexamples which once for all invalidated the supposedly positive solution of Plemelj (1908). This was to be the beginning of a brilliant career, which started with an invited lecture at ICM 90 in Zurich, then took him to many countries as an invited professor, in particular to France where he held a permanent invited position, first in Nice, then in Strasbourg, from 1991 until his untimely death in 2003. Andrei Bolibrukh together with his students devoted his efforts mainly to the Riemann-Hilbert problem, making important steps towards his ultimate goal, which was to find full necessary and sufficient conditions for given monodromy data to be those of a Fuchsian differential system. This led him to work on related questions as well, such as the Birkhoff normal form problem or isomonodromic deformations, on which he published important results too. The first two contributions of this volume, by Y. Ilyashenko and by C. Sabbah, describe some of Bolibrukh's main results. For further biographical information about Andrei Bolibrukh, we refer to the special issue of *Russian Mathematical Surveys* dedicated to our colleague, in particular to the articles [1], [2] and [3], and to our article [4] in the French Mathematical Society's *Gazette*.

In November 2004, a conference – originally planned by Andrei Bolibrukh in Moscow – was organized in Strasbourg, France, as part of the French-Russian program PICS jointly financially supported by the French CNRS (Centre National de la

Recherche Scientifique) and Russian RFBR (Russian Foundation for Basic Research). It was thankfully dedicated to the memory of our late colleague. About one hundred participants from all over the world, including an important delegation from Russia conducted by Academician D. V. Anosov, and another one from Japan, met for a few days of intense activity and discussions. Andrei Bolibrukh would have been particularly pleased with the high number of young mathematicians who attended his conference, including his former students from Russia and from France.

C. Mitschi and C. Sabbah

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Andrey Bolibrukh in 2000

Andrey Bolibrukh with his family in Moscow, 1999

Andrey Bolibrukh with Michael Singer and Vladimir Kostov in Groningen, 1995

Andrey Bolibrukh with Vladimir Arnold and Sofia Kharlamova in Wangenbourg (Alsace), 1999

Table of Contents

Realization of irreducible monodromy by Fuchsian systems and reduction to the Birkhoff standard form (by Andrey Bolibrukh)

Yulij Ilyashenko

Cornell University, Ithaca, New York, U.S.A.
Moscow State and Independent Universities, Steklov Mathematical Institute, Moscow, Russia

Abstract. Two classical results of Bolibrukh are exposed. We try to present clearly the main ideas that are parallel in both proofs, and reduce to a minimum the technical part.

Contents

1 Results

The first result provides a sufficient condition for the solvability of the Riemann–Hilbert problem for Fuchsian systems. This condition is *irreducibility of the monodromy*. The second result provides a sufficient condition for holomorphic equivalence of a linear system near an irregular singular point to the so-called Birkhoff standard form. Again, this condition is *irreducibility of the system.*

Let us pass to more detailed statements.

(*)The author was supported by part by the grants NSF 0400495, RFBR 02-02-00482.

Monodromy data. Consider m points $a_1, \dots, a_m$ on the Riemann sphere, distinct from ∞, and m non-singular linear operators $G_1, \dots, G_m$. The collection

$$(a_1, \dots, a_m, G_1, \dots, G_m) \tag{1}$$

is called *a monodromy data*. The Riemann–Hilbert problem is to construct a Fuchsian system

$$\dot{z} = \sum_1^m \frac{A_j}{t - a_j} z, \quad z \in \mathbb{C}^n, \tag{2}$$

so that a circuit around a_j along a simple loop γ_j produces the monodromy transformation G_j. Strictly speaking the loops γ_j around a_j having the same initial point t_0 should be included in the monodromy data but this detail is hidden in the Plemelj theorem below.

The monodromy group $G = \mathrm{Gr}(G_1, \dots, G_m)$ is called *irreducible* provided that there is no nontrivial subspace of $\mathbb{C}^n$ invariant under all the maps G_j simultaneously. The corresponding monodromy data is called *irreducible* as well.

Theorem 1 ([B1, K]). *Any irreducible monodromy data* (1) *may be realized by a Fuchsian system* (2).

The second problem is *reduction to the Birkhoff standard form*. This is a local problem. Consider a system with non-Fuchsian singular point 0:

$$\dot{z} = \frac{A(t)}{t^r} z, \quad z \in \mathbb{C}^n, \tag{3}$$

where $A(0) \neq 0$, and A is a holomorphic matrix function near zero. A Birkhoff standard form, by definition, is

$$\dot{w} = \frac{B_{r-1}(t)}{t^r} w, \quad w \in \mathbb{C}^n, \tag{4}$$

where B_{r-1} is a matrix polynomial of degree at most $r - 1$. Systems (3) and (4) are called *equivalent* provided that there exists a holomorphic map H of a neighborhood of zero to $\mathrm{GL}(n, \mathbb{C})$ such that the transformation: $w = H(t)z$ takes the system (3) to (4).

A system (3) is *reducible* provided that its matrix $A(t)$ has a form

$$A(t) = \begin{pmatrix} j \times j & * \\ 0 & * \end{pmatrix} \tag{5}$$

or the system (3) is equivalent to a system with such a matrix.

Theorem 2 ([B2]). *An irreducible system* (3) *is equivalent to a Birkhoff standard form* (4).

In order to show the parallel schemes of the proofs we need to recall some classical results together with a *permutation lemma* due to Bolibrukh.

2 Preliminaries

Plemelj's theorem ([P]). *Any monodromy data* (1) *may be realized by a linear system*

$$\dot{z} = A(t)z, \quad z \in \mathbb{C}^n, \tag{6}$$

with a matrix A *holomorphic outside of the* a_j, *having simple poles at all* a_j *and such that* ∞ *is a regular singular point for* (6).

Riemann–Fuchs theorem ([AI]). *A fundamental matrix of a linear system near a regular singular point at infinity has the form*

$$Z(t) = Mt^A$$

for some meromorphic matrix function M *with a pole at infinity, and some constant matrix* A. *Without loss of generality* A *may be taken lower triangular.*

A matrix function $\Gamma \colon \mathbb{C} \to \mathrm{GL}(n, \mathbb{C})$ is called a *monopole* at infinity provided that it is polynomial and $\det \Gamma = \text{ const } \neq 0$. Example:

$$\Gamma = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Notation. In what follows, H_a denotes a holomorphic matrix function $(\widehat{\mathbb{C}}, a) \to \mathrm{GL}(n, \mathbb{C})$. In particular, $H_a(a)$ is invertible. Let M_a be a meromorphic matrix function in a punctured neighborhood of a with values in $\mathrm{GL}(n, \mathbb{C})$ and a pole at a. We will use H_0, H_∞, M_0, M_∞.

Sauvage lemma ([H]). *Any matrix* $M = M_\infty$ *meromorphic at infinity may be decomposed in a product*

$$M_\infty = \Gamma t^K H_\infty, \tag{7}$$

where Γ *is a monopole,* H_∞ *is as above, and* K *is a diagonal matrix with entries* $k_1 \geq k_2 \geq \cdots \geq k_n$. *For such matrices we will write for brevity:* $K \searrow$. *The diagonal elements of the matrix* K *are called "the splitting indices of* M_∞*".*

Birkhoff–Grothendieck theorem. *Any holomorphic invertible matrix function* M *in a ring* $1 \leq |t| \leq 2$ *may be decomposed in a product*

$$M = H_0 M_\infty,$$

where H_0 *is defined for* $|t| \leq 2$, M_∞ *is defined for* $|t| \geq 1$.

Permutation lemma (Bolibrukh). *For any matrix* $K \searrow$ *and matrix function* H_∞ *there exist a diagonal matrix* K_1 *with the entries of* K, *but permuted, and a matrix* $\tilde{H}_\infty$ *such that*

$$t^k H_\infty = \tilde{H}_\infty t^{K_1}.$$

3 Schemes of the proofs

After these preparations we can describe the schemes of the proofs of Theorems 1 and 2, see Figure 1 and Figure 2 below.

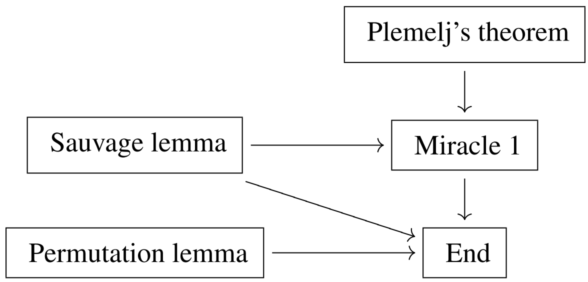

Figure 1. Scheme of the proof of the Bolibrukh–Kostov theorem.

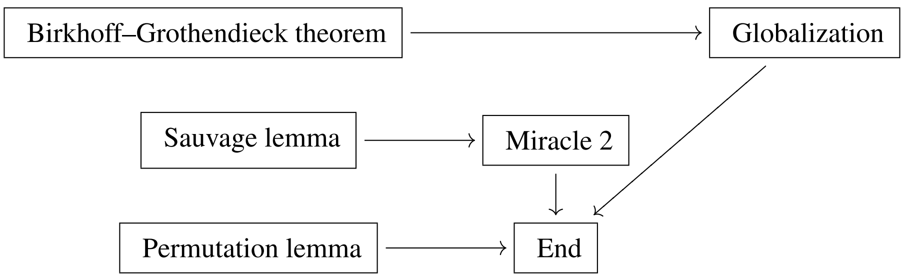

Figure 2. Scheme of the proof of Theorem 2.

The lower parts of both figures are not occasionally the same. In fact, after the miracle happens, the proof in both cases follows absolutely the same lines.

4 Some sufficient conditions for a singular point to be Fuchsian

Below we provide several conditions for a non-univalent matrix function Y with a logarithmic branch-point at infinity to be a fundamental matrix of a Fuchsian system $\dot{Y} = CY$. The conditions are stated and checked in the same way: the matrix $\dot{Y}Y^{-1}$ should be holomorphic and have a simple zero at infinity.

Lemma 1. *The following are fundamental matrixes of Fuchsian systems near infinity:*

$$Y = t^A, \tag{i}$$

$$Y = H_\infty t^A, \tag{ii}$$

$$Y = t^N t^A, \quad N \searrow, A \text{ is lower triangular}, \tag{iii}$$

$$Y = H_\infty Y_\infty, \tag{iv}$$

where Y_∞ is a fundamental matrix of a Fuchsian system at infinity.

Proof. The proof is straightforward: in all the four cases it is easy to check that $\dot{Y}Y^{-1}$ is holomorphic with a zero at infinity. If the zero is of order higher than one, then infinity is a regular point for Y. □

5 Miracle one

In this section we present the key ingredient of the proof of the Bolibrukh–Kostov theorem. We are now in the first line of Figure 1.

Let (1) be the irreducible monodromy data, (6) be the same system as in Plemelj's theorem and Z be a fundamental matrix for system (6).

By the Riemann–Fuchs theorem, Z may be decomposed at infinity as

$$Z = M_\infty t^A. \tag{8}$$

Without loss of generality, we may assume that A is lower triangular. Now we pass to the second line in Figure 1.

Lemma 2. *In the assumption above, let* (7) *be the decomposition from the Sauvage lemma for M_∞. Then the splitting indexes of M_∞ satisfy the inequality*

$$k_j - k_{j+1} < m. \tag{9}$$

Proof. We will prove that if (9) is violated then system (6), hence monodromy (1), is reducible. The main tool to prove this is the well-known statement: The total number of poles of a meromorphic function on the Riemann sphere is equal to the total number of its zeros. This statement will be applied to a fundamental matrix W of a new system constructed as follows.

Let Γ be as in (7), and

$$W = \Gamma^{-1} Z, \quad B = \dot{W} W^{-1} = (b_{ij}). \tag{10}$$

Then B has m simple poles at $a_1, \ldots, a_m$, and no other poles in $\mathbb{C}$. Hence, all elements $b_{ij}(t)$ have a zero of order no greater than m at ∞, or vanish identically. Now, near infinity,

$$B = \dot{W} W^{-1} = \left(\frac{K}{t} W + t^K C(t) t^{-K} \right), \tag{11}$$

where

$$C(t) = \dot{Y}Y^{-1}, \quad Y = H_\infty t^A. \tag{12}$$

By the second condition from Lemma 1, the matrix Y is fundamental for a Fuchsian system at ∞. Hence, $C(\infty) = 0$. Let us now suppose that $k_j - k_{j+1} \geq m$ for some j. Then

$$b_{j+1,j}(t) = c_{j+1,j}(t)k^{k_{j+1}-k_j}.$$

The right hand side has a zero of order at least $m+1$ at infinity. Hence, $b_{j,j+1} \equiv 0$.

Let now $l \leq j, i \geq j+1$. Then $k_i - k_l \leq k_{j+1} - k_j \leq -m$. Hence, for the same reason as above, $b_{il} \equiv 0$. Therefore, the lower corner to the left of b_{jj} and below is zero, see (5). The system $\dot{W} = BW$ is reducible; hence, the equivalent system (6) also is, a contradiction. □

Before ending the proof of the Bolibrukh–Kostov theorem, we will show the second miracle.

6 Miracle two

We begin with globalization thus being in the first line of Figure 2. Let Z_1 be a fundamental matrix of system (3). Then

$$Z_1 = \Phi t^A$$

for some constant matrix A, and a holomorphic invertible Φ defined in a punctured neighborhood of zero; Φ may have an essential singularity at 0. Rescaling t we may assume that Φ is defined in the ring $R : 1 \leq |t| \leq 2$. By the Birkhoff–Grothendieck theorem,

$$\Phi = H_0 M_\infty. \tag{13}$$

Let

$$Z = \begin{cases} H_0^{-1} Z_1 & \text{for } |t| \leq 2, \\ M_\infty t^A & \text{for } |t| \geq 1. \end{cases} \tag{14}$$

Factorization (10) guarantees that two different representations of Z agree in K. Hence, Z is a fundamental matrix of a system equivalent to (3) near zero but now defined on all the Riemann sphere.

Remark. Let us now observe that the Birkhoff standard form is globally defined on the Riemann sphere as well and has a Fuchsian singular point at infinity. Conversely, a linear system with a non-Fuchsian singular point at zero, Fuchsian singular point at infinity and no other singular points, has a Birkhoff standard form at zero. Our goal now is to replace Z from (14) by a new fundamental matrix W which is equivalent to Z at 0 and has a Fuchsian singular point infinity.

We now pass to the second line in Figure 2. Let (7) be a Sauvage decomposition for M_∞, where M_∞ is the same as in (9).

Lemma 3. *With the above assumptions, the consecutive splitting indexes of the matrix* M_∞ *differ by less than* r.

Proof. The proof follows the same lines as for Lemma 2. Suppose the converse is true. Let

$$k_j - k_{j+1} \geq r \quad \text{for some } j. \tag{15}$$

Let $W = \Gamma^{-1} Z$ where Γ is from (7). Let $Y = H_\infty t^A$. Consider $B = \dot{W} W^{-1}$. On one hand $B = (b_{ij})$ has a pole of order r at 0; hence, b_{ij} cannot have a zero of order greater than r at infinity, or else, b_{ij} vanishes identically.

Now, near infinity, B has the form (11) with Y from (12). As in Lemma 2, we prove that under assumption (15) $b_{il} \equiv 0$ for $i > j$, $l \leq j$. Thus, the system with matrix (11) is reducible. Hence, the original system equivalent to the system $\dot{W} = BW$ is reducible itself, a contradiction. □

7 End of the proofs

We now pass to the bottom line in Figure 1, and conclude the proof of Theorem 1.

Let W be the same as in (10). Take a matrix $N \searrow$ such that for any j, $n_j - n_{j+1} > (n-1)(m-1)$. Then this difference is larger than the difference between any two elements of the matrix K from (11), subject to restriction (9). If K_1 is a diagonal matrix with diagonal elements of K permuted, then $N + K_1 \searrow$.

Consider now the following representation for the original fundamental matrix Z near infinity:

$$Z = M_\infty t^A = M_\infty t^{-N} t^N t^A = \tilde{M} Y, \tag{16}$$

where $\tilde{M} = M_\infty t^{-N}$, $Y = t^{-N} t^A$. By statement (iii) of Lemma 1, Y is Fuchsian at infinity. Let now

$$\tilde{M} = \Gamma t^K H_\infty.$$

Then H_∞ is Fuchsian, and Lemma 2 is applicable. Hence, for the matrix K, restriction (9) holds. By the Permutation lemma,

$$t^K H_\infty = \tilde{H}_\infty t^{K_1},$$

where K_1 is a diagonal matrix with diagonal elements of K permuted. Hence,

$$W = \Gamma \tilde{H}_\infty \tilde{Y}, \quad \tilde{Y} = t^{N+K_1} t^A,$$

where $N + K_1 \searrow$ and A is lower triangular. Therefore, the matrix $\tilde{Y}$ is Fuchsian at infinity; hence, $H_\infty \tilde{Y}$ is, by statement (iv) of Lemma 1. Therefore, the matrix $V = \Gamma^{-1} W$ has Fuchsian singular points at $a_1, \ldots, a_m$, and at infinity as well. This is the sought Fuchsian realization of the monodromy data. Theorem 1 is proved.

Let us now prove Theorem 2. We are in the bottom line of Figure 2, and the arguments are quite the same. We only take N with the gap condition $n_j - n_{j+1} > (n-1)(r-1)$.

Let now Z be the same as in (14). Reproducing word by word the previous arguments beginning with formula (16), we prove that the matrix $V = \Gamma^{-1}W$ is Fuchsian at infinity. This matrix is equivalent to Z_1 near 0, and the corresponding system for V has a Birkhoff standard form by the remark from Section 6. This concludes the proof of Theorem 2.

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The work of Andrey Bolibrukh on isomonodromic deformations

Claude Sabbah

UMR 7640 du CNRS
Centre de Mathématiques Laurent Schwartz
École polytechnique
91128 Palaiseau cedex, France
email: `sabbah@math.polytechnique.fr`

Abstract. We give a description of the work of Andrey Bolibrukh on isomonodromic deformations and relate it to existing results in this domain.

Contents

Introduction

Let me begin by quoting [8]:

> "*Therefore, in essence, the invariant geometric language of vector bundles is unavoidable for the rigorous analysis of the inverse monodromy problem and isomonodromy deformations in the case of general linear systems. At the same time, for specific linear systems related to the Painlevé equations, it is possible to perform a rigorous study of the inverse problem on the basis of analytic considerations only.*"

This phrase illustrates the approach of Andrey to differential equations: he was able to speak the language of vector bundles with algebraic/differential geometers and the classical language with analysts. His work on isomonodromy problems was much influenced by the algebraic geometry approach, through Malgrange's papers, but he was also much involved in applications to Painlevé equations from a more analytic and concrete point of view, working on explicit formulas, as in [8]. For instance, he worked on an algorithmic computation of the τ function of Miwa and Jimbo in [7].

In this article, we try to explain his results concerning isomonodromic deformations of systems with regular singularities or Fuchsian systems[1] and to relate them to other results in this domain. When necessary, we use the language of vector bundles with connection and, in any case, we try to give translations of the results in this language.

1 What is an isomonodromic deformation?

Let X_t be a holomorphic family of connected complex manifolds parametrized by a complex connected manifold T with base point t^o, which have constant fundamental group $\pi_1(X_t, *)$. Let (E_{t^o}, ∇_{t^o}) be a vector bundle on X_{t^o} equipped with a flat holomorphic connection $\nabla_{t^o} : E_{t^o} \to \Omega^1_{X_{t^o}} \otimes_{\mathcal{O}_{X_{t^o}}} E_{t^o}$.

An isomonodromic deformation of (E_{t^o}, ∇_{t^o}) is a holomorphic family E_t of holomorphic vector bundles equipped with a *flat* connection $\nabla_t : E_t \to \Omega^1_{X_t} \otimes_{\mathcal{O}_{X_t}} E_t$ such that the conjugation class of the monodromy representation defined by horizontal sections of ∇_t is constant.

Such a situation often occurs in the following way. We start with a complex manifold $\overline{X}$ with a smooth divisor Y and a holomorphic map $\pi : \overline{X} \to T$ which is assumed to be smooth on the pair $(\overline{X}, Y)$ and therefore defines a C^∞ fibration. We put $X = \overline{X} \smallsetminus Y$ and π still defines a C^∞ fibration on X, so that all fibres X_t have the same topological type (in fact, the topological type of the pair $(\overline{X}_t, Y_t)$ is constant). Assume that we have a holomorphic vector bundle E on $\overline{X}$ and a *flat meromorphic* connection ∇ on E with poles along Y. The π_1 of the fibers X_t is constant, and the monodromy representation of $\pi_1(X_t)$ defined by each ∇_t on $E_{|X_t}$ is constant up to conjugation.

Exemple 1.1. We will mainly consider below the case where T is the universal cover of the n-fold product $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ minus diagonals (a point $a \in T$ lies over an ordered set of n distinct points $a_1, \dots, a_n$ of $\mathbb{P}^1$), $\overline{X} = \mathbb{P}^1 \times T$ and $Y = \bigcup_{i=1}^n Y_i$, where $Y_i = \{x = a_i\}$ and where x is the component in the first $\mathbb{P}^1$. Then X is the $n+1$-fold product $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ minus diagonals and the fibre X_a is $\mathbb{P}^1 \smallsetminus \{a_1, \dots, a_n\}$.

We will also consider the case where $\overline{X} = D \times T$, where D is a disc centered at 0 in $\mathbb{C}$, and $Y = \{0\} \times T$.

[1] This explains why we do not consider his joint article [8], which is concerned with a special example of irregular isomonodromic deformation and would necessitate the introduction of many other notions.

One has to be careful: the integrability property above (flatness of ∇) may be strictly stronger than isomonodromy, if one does not impose a supplementary condition, namely that ∇ has *regular singularity* along Y. In practice however, various authors use the word "isomonodromy" instead of "integrability" when irregular singularities occur. This is justified by an extension of what one calls "monodromy representation": in the irregular singular case, one adds the Stokes data to the classical monodromy representation.

In [5], [3], Andrey considered the question of comparing precisely these two notions, namely isomonodromy and integrability. Although he only considers the example above, his results apply in a much more general situation.

Theorem 1.2 (mainly in [3] and [5]). *Let $\pi : \overline{X} \to T$ and $Y \subset \overline{X}$ be as above. Assume that the π_1 of some (or any) fibre X_t is finitely generated.*

(1) *Let (E_{t^o}, ∇_{t^o}) be a vector bundle on $\overline{X}_{t^o}$ equipped with a meromorphic flat connection with poles along Y_{t^o}, having regular singularities along Y_{t^o}. Then any isomonodromic deformation of $(E_{t^o|X_{t^o}}, \nabla_{t^o})$ in the family $\pi : X \to T$ can be realized, locally near t^o, by a meromorphic bundle $E'(*Y)$ equipped with a* flat *connection ∇' with regular singularities along Y such that $(E'_{t^o}(*Y_{t^o}), \nabla'_{t^o}) \xrightarrow{\sim} (E_{t^o}(*Y_{t^o}), \nabla_{t^o})$.*

(2) *Let $(E, \nabla_{\overline{X}/T})$ be a vector bundle on $\overline{X}$ equipped with a meromorphic relative flat connection $\nabla_{\overline{X}/T}$ with poles along Y, defining an isomonodromic deformation on X and such that each (E_t, ∇_t) has regular singularities along Y_t. Then, locally on T, there exists a meromorphic connection ∇ on E with poles on Y, having $\nabla_{\overline{X}/T}$ as associated relative connection, and with regular singularities along Y.*

Remarks 1.3. (1) Of course, the assumption on the fundamental group of fibres is satisfied in all usual examples.

(2) Assume that each fibre X_t is a curve. If we moreover assume that (E_{t^o}, ∇_{t^o}) is *Fuchsian* (*i.e.*, has only simple poles at each point of Y_{t^o}), then there exists locally on T a unique isomonodromic deformation (E, ∇) where ∇ has logarithmic poles along Y (*cf.* [11] or Theorem 3.1 below). There may exist other isomonodromic deformations. By Theorem 1.2, these deformations can be searched as meromorphic connections with regular singularities along Y. This will be used in § 2.

(3) Let us explain the difference between the two statements. In the second one, we fix a meromorphic structure of the bundle along Y and, knowing that the relative connection is meromorphic with respect to it, we show that the absolute connection is also meromorphic with respect to this structure. In the first one, such a structure is constructed simultaneously with the absolute connection, in such a way that the latter is meromorphic with respect to the former.

Proof. For the first part, the proof has 2 steps: the smooth step, where one forgets about the polar locus Y and the meromorphic step, where one shows that ∇ can be chosen to be meromorphic along Y.

Proof of 1.2 (1)*, first step.* Consider a holomorphic vector bundle E on X equipped with a flat relative connection $\nabla_{X/T}: E \to \Omega^1_{X/T} \otimes_{\mathcal{O}_X} E$. By the Cauchy–Kowalevski theorem with parameters, $\ker \nabla_{X/T}$ is a locally constant sheaf of locally free $\pi^{-1}\mathcal{O}_T$-modules (*cf.* [10, Théorème 2.23]) and $(E, \nabla_{X/T}) \xrightarrow{\sim} (\mathcal{O}_X \otimes_{\pi^{-1}\mathcal{O}_T} \ker \nabla_{X/T}, d_{X/T} \otimes \mathrm{id})$. Under the isomonodromy condition, we want to show that there exists, locally with respect to T, a locally constant sheaf $\mathcal{F}$ of finite dimensional $\mathbb{C}$-vector spaces on X such that $\ker \nabla_{X/T} = \pi^{-1}\mathcal{O}_T \otimes_{\mathbb{C}} \mathcal{F}$. We will then define ∇ so that $(E, \nabla) \xrightarrow{\sim} (\mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{F}, d_X \otimes \mathrm{id})$, and by definition we will have $(E_{t^o}, \nabla_{t^o}) \xrightarrow{\sim} \big(\mathcal{O}_{X_{t^o}} \otimes_{\mathbb{C}} \mathcal{F}_{|X_{t^o}}, d\big)$.

We fix $t^o \in T$ and work in a neighbourhood of t^o. The fundamental group $\pi_1(X_{t^o}, *)$ is generated by loops $\gamma_1, \dots, \gamma_p$. A linear representation of $\pi_1(X_{t^o}, *)$ in $\mathrm{GL}_d(\mathbb{C})$ consists of p invertible matrices $M_1, \dots, M_p$ which satisfy the same relations as the γ_i do. The set Rep of these data is therefore the closed algebraic subset of $(\mathrm{GL}_d(\mathbb{C}))^p$ defined by algebraic equations of the form $M_1^{n_1} \cdots M_p^{n_p} - \mathrm{id} = 0$. The group $\mathrm{GL}_d(\mathbb{C})$ acts on $(\mathrm{GL}_d(\mathbb{C}))^p$ by $P \cdot (M_1, \dots, M_p) = (PM_1P^{-1}, \dots, PM_pP^{-1})$ and leaves Rep invariant. The orbit of a given representation ρ^o consists of the representations that are conjugate to ρ^o.

The assumption of the theorem shows that there exists a neighbourhood V of t^o in T and a holomorphic map $V \to (\mathrm{GL}_d(\mathbb{C}))^p$, sending t^o to ρ^o, such that its image is contained in the orbit of ρ^o. As the natural map $\mathrm{GL}_d(\mathbb{C}) \to \mathrm{GL}_d(\mathbb{C}) \cdot \rho^o$ has everywhere maximal rank, one can locally lift $V \to \mathrm{GL}_d(\mathbb{C}) \cdot \rho^o$ to a holomorphic map $V \to \mathrm{GL}_d(\mathbb{C})$. The holomorphic family ρ_t of representations of $\pi_1(X_{t^o}, *)$ is therefore conjugate to the constant family ρ_o.

Proof of 1.2 (1)*, second step.* By [10], the bundle $\mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{F}$ with its flat connection d extends as a meromorphic bundle $E'(*Y)$ with a connection ∇' having regular singularities along Y. The isomorphism $(E_{t^o}, \nabla_{t^o}) \xrightarrow{\sim} (E'_{|X_{t^o}}, \nabla'_{t^o})$ that we constructed in the first step is a relative horizontal section σ on X_{t^o} of the meromorphic bundle $\mathcal{H}om_{\mathcal{O}_{\overline{X}_{t^o}}(*Y_{t^o})}(E_{t^o}(*Y_{t^o}), E'_{t^o}(*Y_{t^o}))$. The connection on this bundle (obtained from ∇_{t^o} and ∇'_{t^o}) has a regular singularity along Y_{t^o}, hence the section σ is meromorphic along Y_{t^o}. A similar argument applies to σ^{-1}, so that we have an isomorphism $(E_{t^o}(*Y_{t^o}), \nabla_{t^o}) \xrightarrow{\sim} (E'_{|\overline{X}_{t^o}}(*Y_{t^o}), \nabla'_{t^o})$.

Proof of 1.2 (2). The proof is similar to that of 1.2 (1). We first construct $(E'(*Y), \nabla')$ as above. We now have an isomorphism $(E_{|X}, \nabla) \xrightarrow{\sim} (E'_{|X}, \nabla')$, as constructed in the first step: it is a horizontal section σ on X of the meromorphic bundle $\mathcal{H}om_{\mathcal{O}_{\overline{X}}(*Y)}(E(*Y), E'(*Y))$. Restricting to each fibre $\overline{X}_t$, the connection on this bundle (obtained from $\nabla_{\overline{X}/T}$ and $\nabla'_{\overline{X}/T}$) has a regular singularity along Y_t, hence the section $\sigma_{|X_t}$ is meromorphic along Y_t. The order of its pole is locally bounded by a constant computed from the matrices of $\nabla_{\overline{X}/T}$ and $\nabla'_{\overline{X}/T}$ in local meromorphic bases of $E(*Y)$ and $E'(*Y)$. Hence σ is meromorphic along Y. A similar argument applies to σ^{-1}. We therefore have an isomorphism $(E(*Y), \nabla_{\overline{X}/T}) \xrightarrow{\sim} (E'(*Y), \nabla'_{\overline{X}/T})$. $\square$

2 Isomonodromic deformations: the local setting

In this section, we consider a disc D centered at the origin in the complex plane, with coordinate x, and a parameter space T, which is a neighbourhood of the origin in $\mathbb{C}^n$, with coordinates $t = (t_1, \dots, t_n)$.

We consider a linear differential system of size d in the variable x, which is Fuchsian, that is

$$x \cdot \frac{du}{dx} = A(x) \cdot u(x), \tag{$*$}$$

where $u(x)$ is a vector of size d of unknown functions, and $A(x)$ is a matrix of size d with holomorphic entries.

In other words, we consider the trivial holomorphic vector bundle E^o (free $\mathcal{O}_D$-module) of rank d on D, with a meromorphic connection $\nabla\colon E^o \to \Omega^1_D(\log\{0\}) \otimes_{\mathcal{O}_D} E^o$ having a possible simple pole at the origin.

A Fuchsian isomonodromic deformation of $(*)$ parametrized by T is a system

$$x \cdot \frac{du(x,t)}{dx} = A(x,t) \cdot u(x,t), \tag{$*_t$}$$

such that $A(x,t)$ is holomorphic and that, for any $t^o \in T$, the monodromy at the origin of the system $(*_{t^o})$ is independent of t^o (up to conjugation). By Theorem 1.2, the isomonodromy condition can also be stated by saying that there exists a matrix

$$\Omega = A(x,t)\,\frac{dx}{x} + \sum_{i=1}^{n} \Omega_i(x,t)\,dt_i, \tag{2.1}$$

where the $\Omega_i(x,t)$ are holomorphic on $D^* \times T$, such that Ω satisfies the integrability condition

$$d\Omega + \Omega \wedge \Omega = 0.$$

We say that the isomonodromic deformation is *regular* if Ω is *meromorphic* along $x = 0$ (equivalently, if each Ω_i is so). By Theorem 1.2 (2), any Fuchsian isomonodromic deformation is regular.

Equivalently, we are given a trivial vector bundle E on $D \times T$ with an *integrable* meromorphic connection

$$\nabla\colon E \to \Omega_{D\times T}\big[*(\{0\}\times T)\big] \otimes_{\mathcal{O}_{D\times T}} E$$

having possible poles along $x = 0$ only, and such that its restriction to each $D \times \{t^o\}$ is logarithmic. In particular, the meromorphic connection ∇ has *regular singularities* along $x = 0$. But it may or may not be *logarithmic* along $x = 0$, that is, some Ω_i may not be *holomorphic* at $x = 0$. In the following, we consider only Fuchsian isomonodromic deformations, even if some statements hold in a more general situation.

Proposition 2.2. *In an isomonodromic deformation, the eigenvalues of $A(0,t)$ are independent of t.*

Proof. Indeed, for t fixed, the characteristic polynomial of $A(0,t)$ determines the characteristic polynomial of the monodromy of the corresponding system by the following rule: to any term $(X-\alpha)^{\mu_\alpha}$, associate $(S-e^{-2i\pi\alpha})^{\mu_\alpha}$. As the latter is constant by isomonodromy, the eigenvalues of $A(0,t)$ can only vary by integral jumps, so, by continuity, they are constant. □

Giving a Fuchsian system of rank d on the disc D, with pole at 0 only, is equivalent to giving a $\mathbb{C}$-vector space H of dimension d equipped with an automorphism M (monodromy) and a decreasing filtration $F^\bullet H$ stable by M (called the *Levelt filtration*). This filtration takes into account the resonances (nonzero integral differences of eigenvalues) in the matrix of the connection.

Corollary 2.3 ([3, Theorem 2]). *In an isomonodromic deformation, the Levelt normal form can be achieved locally holomorphically with respect to the parameters.*

Proof. According to the previous proposition, there exists, locally with respect to t (say near t^o), a base change such that the matrix $A(0,t)$ is block-diagonal, one (constant) eigenvalue per block. We order the blocks in such a way that the integral parts of the eigenvalues are decreasing, hence we get a diagonal matrix $\Lambda = \operatorname{diag}(\lambda_1,\dots,\lambda_d)$ where the integers λ_j satisfy $\lambda_j \geqslant \lambda_{j+1}$. let us put $\lambda = \max_{i,j}\left|\lambda_i-\lambda_j\right|$. Then $A_0(t) := A(0,t)$ commutes with Λ. One looks for a formal power series $\hat{P}(x,t) = \mathrm{id} + \sum_{k\geqslant 1} x^k P_k(t)$ and matrices $B_1(t),\dots,B_\lambda(t)$ such that $[\Lambda, B_j(t)] = -jB_j(t)$ ($j=1,\dots,\lambda$) and, letting $B(x,t) = A_0(t) + xB_1(t) + \cdots + x^\lambda B_\lambda(t)$, we have

$$x\hat{P}'(x,t) = \hat{P}(x,t)B(x,t) - A(x,t)\hat{P}(x,t).$$

The formal solution P is obtained by solving successively, for $k \geqslant 1$,

$$(\operatorname{ad} A_0(t) + k\,\mathrm{id})P_k(t) = B_k(t) + \Phi_k(A_{\leqslant k}(t), B_{<k}(t), P_{<k}(t))$$

where Φ_k depends on the previous coefficients. As $A_0(t)$ commutes with Λ, we can decompose this equation on the eigenspaces of the semisimple endomorphism $\operatorname{ad}\Lambda$. For the eigenvalue μ, we have then to solve, for $k \geqslant 1$,

$$(\operatorname{ad} A_0(t) + k\,\mathrm{id})P_k^{(\mu)}(t) = \begin{cases} B_k(t) + \Phi_k^{(\mu)}(A_{\leqslant k}(t), B_{<k}(t), P_{<k}(t)) & \text{if } \mu + k = 0,\\ \Phi_k^{(\mu)}(A_{\leqslant k}(t), B_{<k}(t), P_{<k}(t)) & \text{if } \mu+k \neq 0,\end{cases}$$

by assumption on B_k. If $k \neq -\mu$, the endomorphism $(\operatorname{ad} A_0(t) + k\,\mathrm{id})$ is invertible on $\ker(\operatorname{ad}\Lambda - \mu\,\mathrm{id})$, hence we can solve in a unique way the second line. If $k=-\mu$, we choose $B_k(t)$ so that the right-hand term is in the image of $(\operatorname{ad} A_0(t) + k\,\mathrm{id})$.

Then, by a standard argument for regular singularities, one shows that the matrix $\hat{P}$ is convergent in some neighbourhood of $(0,t^o)$ and we denote it by P. Therefore, after the base change given by P, the matrix of the connection can be written as

$$\Omega' = \big(A_0(t) + xB_1(t) + \cdots + x^\lambda B_\lambda(t)\big)\frac{dx}{x} + \sum_{i=1}^n \Omega_i'(x,t)\,dt_i \tag{2.4}$$

with $\Omega_i'(x,t)$ meromorphic and having a pole of order less than or equal to that of Ω_i along $x=0$, as the base change is holomorphically invertible. □

In terms of filtrations, this result means that the family $F_t^\bullet H$ of filtrations of H parametrized by T is holomorphic, *i.e.*, defines a filtration of the bundle $\mathcal{O}_T \otimes_{\mathbb{C}} H$ by holomorphic subbundles, in such a way that the graded pieces are vector bundles (*i.e.*, the rank does not jump with t).

Corollary 2.5 ([3, Theorem 3]). *In an isomonodromic deformation, the pole of each matrix Ω_i along $x=0$ has order at most λ.*

Proof. By Corollary 2.3, we can assume that we start with a matrix Ω as in (2.1) such that $A(x,t)$ has the Levelt normal form (2.4). The eigenvalues of $A(0,t)-\Lambda$ do not differ by a nonzero integer and the monodromy matrix is $\exp(-2i\pi(A(0,t)-\Lambda))$. Then there exists a holomorphic invertible matrix $C(t)$ such that $\exp(-2i\pi(A(0,t)-\Lambda)) = C(t)^{-1}\cdot\exp(-2i\pi(A(0,0)-\Lambda))\cdot C(t)$. So, after the base change of matrix $C(t)$, the connection can be written as $d+(A(0,0)-\Lambda)dx/x$. Setting $P(x,t)=x^\Lambda C(t)$, we therefore have $\Omega_i = x^\Lambda \partial_{t_i} C(t)C(t)^{-1}x^{-\Lambda}$, which has a pole of order $\leqslant \lambda$ along $x=0$. □

This corollary implies in particular that, under some circumstances, any regular isomonodromic deformation is in fact logarithmic. This occurs for instance when $A(0)$ in $(*)$ is nonresonant, that is, if its eigenvalues do not differ by a nonzero integer.

3 Logarithmic isomonodromic deformations of Fuchsian systems on the Riemann sphere: the Schlesinger system

3.a The Painlevé property, after Malgrange

Let us recall the proof of the Painlevé property of the Schlesinger system given by Malgrange in [11].

We fix a finite set of distinct points $a^o = \{a_1^o,\dots,a_n^o\}$ on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ and we consider a vector bundle E^o on $\mathbb{P}^1$ equipped with a connection ∇^o having logarithmic poles at a^o and no other pole.

Our parameter space T is now global: it is the universal covering of $(\mathbb{P}^1)^n \smallsetminus$ diagonals (one can reduce the dimension by 3, if we fix by a homography three points among the a_i to $0,1,\infty$, say). We view a^o as point of $(\mathbb{P}^1)^n \smallsetminus$ diagonals and we choose a lift $\tilde a^o$ of a^o in T. On $\mathbb{P}^1\times T$ we have natural hypersurfaces Y_i defined by the equation $x=\tilde a_i$ ($\tilde a_i$ is the lift to T of the function a_i).

Theorem 3.1 ([11]). *There exists a unique vector bundle E on $\mathbb{P}^1\times T$ equipped with an integrable logarithmic connection ∇ having poles along the hypersurfaces Y_i, and with an identification $(E,\nabla)_{\mathbb{P}^1\times\{\tilde a^o\}} \xrightarrow{\sim} (E^o,\nabla^o)$.* □

Assume now that E^o is trivial. If we fix a basis of this bundle and a coordinate x on $\mathbb{P}^1 \smallsetminus \{\infty\}$, giving ∇^o is equivalent to giving a Fuchsian system

$$\frac{du}{dx} = \sum_{i=1}^{n} \frac{A_i^o}{x - a_i^o} \cdot u, \tag{3.2}$$

where the A_i^o are $d \times d$ matrices. Then there exists a divisor Θ in T consisting of points $\tilde{a}$ where $E_{\mathbb{P}^1 \times \{\tilde{a}\}}$ is not trivial. More precisely, there exists a meromorphic (along $\mathbb{P}^1 \times \Theta$) trivialization of E which extends the trivialization of E^o. In other words, there exists a basis of $E(*\Theta)$ extending the given basis of E^o. The matrix of ∇ in this basis takes the form

$$\sum_{i=1}^{n} A_i(\tilde{a}) \frac{d(x - \tilde{a}_i)}{(x - \tilde{a}_i)} + \sum_{i=1}^{n} B_i(\tilde{a}_i) d\tilde{a}_i.$$

The basis can moreover be chosen in such a way that all the B-terms vanish identically: this is obtained by imposing flatness with respect to the residual connection on Y_n, say. To simplify notation, it is simpler (but not less general) to assume that $Y_n = \{\infty\} \times T$. In such a basis, the matrix of ∇ thus takes the form

$$\sum_{i=1}^{n} A_i(\tilde{a}) \frac{d(x - \tilde{a}_i)}{(x - \tilde{a}_i)} \tag{3.3}$$

and the integrability property is equivalent to the fact that the A_i are solutions of the *Schlesinger system*

$$dA_i = \sum_{j \neq i} [A_i, A_j] \frac{d(\tilde{a}_i - \tilde{a}_j)}{(\tilde{a}_i - \tilde{a}_j)}, \quad i = 1, \dots, n. \tag{Schl}$$

These equations imply in particular that the residue $-\sum_i A_i(\tilde{a})$ along $\{\infty\} \times T$ is constant (the basis is chosen precisely so that this property is satisfied).

Corollary 3.4. *The solutions of the Schlesinger system* (Schl) *with initial value A_i^o at $\tilde{a}^o$ are meromorphic on T with possible poles along Θ only.* □

The behaviour of the solutions of (Schl) near the polar set Θ is hard to analyze. In [4] and [7], Andrey has given a method to produce examples and describe in concrete terms this behaviour. I will try to explain his approach.

3.b Equation for the "theta divisor"

Starting from a solution of the Schlesinger system with initial values A_i^o at $\tilde{a}^o$, we obtain a hypersurface Θ in T, which is the set of points $\tilde{a}$ where the bundle $E_{\mathbb{P}^1 \times \{a\}}$ is not trivial (however its degree remains equal to 0). Let $\tilde{a}^*$ be a point on Θ. We will work locally near $\tilde{a}^*$, and therefore we will not distinguish between $\tilde{a}$ and $a \in$

$(\mathbb{P}^1)^n \smallsetminus$ diagonals. Moreover, we now forget about the initial data which have produced this isomonodromic deformation and the corresponding Θ, and we denote by T a small neighbourhood of a^*. We also assume, for convenience, that none of the points a_i ($i = 1, \dots, n$ and $a \in T$) is equal to ∞ (it is enough to assume that this is true for a_i^* and take T small enough).

The bundle E_{a^*} is not trivial. By the Birkhoff–Grothendieck theorem, it is decomposed as a sum of rank-one vector bundles $\bigoplus_{j=1}^d \mathcal{O}(k_j)$ with some $k_j \neq 0$ and $k_1 \geqslant \cdots \geqslant k_d$ (and $\deg E_{a^*} = k_1 + \cdots + k_d = 0$, so that there exists $\ell, m \in \{1, \dots, d\}$ such that $k_\ell - k_m \geqslant 2$). The typical example when $d = 2$ is $E_{a^*} = \mathcal{O}(1) \oplus \mathcal{O}(-1)$.

The meromorphic bundle $E[*(\infty \times T)]$ is trivializable. It contains E as a holomorphic subbundle. The bundle with connection (E, ∇) can be characterized as the unique extension of $\big(E[*(\infty \times T)], \nabla\big)$ which is holomorphic at infinity.

On the other hand, there exists a holomorphic subbundle $E_{a^*}^{(0)}$ of the meromorphic bundle $E_{a^*}[*\infty]$ which is trivial and on which the connection ∇ has only logarithmic poles. One can choose $E_{a^*}^{(0)}$ such that, in any trivialization, the matrix of the connection ∇ has residue $-K^{(0)}$ at ∞, with $K^{(0)} := \operatorname{diag}(k_1^{(0)}, \dots, k_d^{(0)})$ (with $k_i^{(0)} = k_i$): Andrey uses Sauvage Lemma to do so; in terms of vector bundles, remark that there exists a basis of $\mathcal{O}(k_j)[*\infty] = \mathcal{O}[*\infty]$ in which the matrix of the differential d has a pole at ∞ only, which is logarithmic with residue $-k_j$; using the splitting of E_{a^*} given by the Birkhoff–Grothendieck theorem, one gets the desired basis of E_{a^*}.

Denote by $B_i^{(0)}(a^*)$ the residue of the connection ∇ at the pole a_i^*. The matrix of the connection in the chosen basis is then written as

$$\sum_{i=1}^n \frac{B_i^{(0)}(a^*)}{x - a_i^*}\, dx.$$

The point at infinity is an apparent (logarithmic) singularity and we have $\sum_i B_i^{(0)}(a^*) = K^{(0)}$.

Apply now Theorem 3.1 starting with $(E_{a^*}^{(0)}, \nabla)$ to construct a holomorphic subbundle $E^{(0)}$ of $E[*(\infty \times T)]$ with a logarithmic connection having poles at $Y_1 \cup \cdots \cup Y_n \cup (\{\infty\} \times T)$ and, maybe after taking a smaller T, choose the canonical trivialization by extending flatly the basis along $\{\infty\} \times T$. In this basis of $E^{(0)}$, the matrix of ∇ is written in the form

$$\sum_i B_i^{(0)}(a)\, \frac{d(x - a_i)}{(x - a_i)}$$

and the $B_i^{(0)}(a)$ satisfy the Schlesinger system.

Lemma 3.5 ([7, Lemma 1]). *There exists $\ell, m \in \{1, \dots, d\}$ such that $k_m^{(0)} - k_\ell^{(0)} \geqslant 2$ and $i \in \{1, \dots, n\}$ such that the (ℓ, m)-entry $B_{i,\ell m}^{(0)}(a)$ does not vanish identically.*

Proof. Otherwise, the base change near $\{\infty\} \times T$ with matrix $x^{K^{(0)}}$ would simultaneously (with respect to a) eliminate the apparent singularity. We would obtain the bundle E as a result, as mentioned above. This would mean that E_a has splitting type $k_d^{(0)},\dots,k_1^{(0)}$ for any $a\in T$. But E_a is known to be trivial for $a\notin\Theta$, a contradiction. □

Lemma 3.6 ([7, § 2]). *Fix $\ell,m\in\{1,\dots,d\}$ such that $k_m^{(0)}-k_\ell^{(0)}\geqslant 2$ and $B_{i,\ell m}^{(0)}(a)\not\equiv 0$. Put $\tau^{(0)}(a)=\sum_i B_{i,\ell m}^{(0)}(a)a_i=0$ and let $\Theta^{(0)}$ be the support of $\operatorname{Div}(\tau^{(0)})$. Then there exists an holomorphic extension $E^{(1)}[*\Theta^{(0)}]$ of $E[*(\infty\times T)\cup\Theta^{(0)}]$ such that, out of $\Theta^{(0)}$,*

(1) *for any $a\in T\smallsetminus\Theta^{(0)}$, the bundle $E^{(1)}[*\Theta^{(0)}]_a$ is trivial,*

(2) *the connection ∇ is logarithmic on $E^{(1)}[*\Theta^{(0)}]$ with poles on $Y_1\cup\dots\cup Y_n\cup(\infty\times T)$ and its residue along $\infty\times T$ is $-K^{(1)}=-\operatorname{diag}(k_1^{(1)},\dots,k_d^{(1)})$ with*

$$\sum_{j=1}^d (k_j^{(1)})^2\leqslant\sum_{j=1}^d (k_j^{(0)})^2-2.$$ □

Notice that these lemmas imply that the stratum of Θ consisting of points $a\in\Theta$ where the splitting type of E_a is the same as that of E_{a^*} is defined by the equations $B_{i,\ell m}^{(0)}(a)=0$ for all $i=1,\dots,n$ and all pairs ℓ,m with $k_m-k_\ell\geqslant 2$.

If $K^{(1)}=0$, then $E^{(1)}[*\Theta^{(0)}]$ coincides with $E[*\Theta^{(0)}]$, by the uniqueness of the extension of $E[*(\{\infty\}\times T)]$ which is smooth along $\{\infty\}\times T$. We therefore have $\Theta\subset\Theta^{(0)}$.

If $K^{(1)}\neq 0$, we are in the situation of Lemma 3.5, except the fact that all the coefficients are meromorphic along $\Theta^{(0)}$ and maybe not holomorphic. Then, applying Lemmas 3.5, we construct the meromorphic function $\tau^{(1)}$ (with poles on $\Theta^{(0)}$ at most) and we define $\Theta^{(1)}$ as the union of the support of $\operatorname{Div}(\tau^{(1)})$ and $\Theta^{(0)}$. We then construct $E^{(2)}[*\Theta^{(1)}]$, etc.

In a finite number of applications of Lemmas 3.5 and 3.6, we get a divisor $\widetilde\Theta$ in T and an extension $\widetilde E[*\widetilde\Theta]$ of $E[*(\infty\times T)\cup\widetilde\Theta]$ on which the connection has no pole along $\{\infty\}\times T$ and which is trivial. In particular, it coincides with E out of $\widetilde\Theta$ and, more precisely, we have $\widetilde E[*\widetilde\Theta]=E[*\widetilde\Theta]$. By definition of Θ, we have the inclusion

$$\Theta\subset\widetilde\Theta.$$

In some sense, this inductive procedure transfers the apparent polar locus $\{\infty\}\times T$ to a polar locus $\widetilde\Theta$ contained in T.

We also have a finite sequence of meromorphic functions $\tau^{(0)},\tau^{(1)},\dots$ ($\tau^{(0)}$ being holomorphic) and we put $\tilde\tau=\prod_\nu\tau^{(\nu)}$.

On the other hand, denote by τ the function of Miwa and Jimbo defining the divisor Θ, after the theorem of Miwa (see [11]). Near a^* we have

$$\frac{d\tau}{\tau}=\frac12\sum_{i\neq j}\operatorname{tr}\big(B_i(a)B_j(a)\big)\frac{d(a_i-a_j)}{(a_i-a_j)}.$$

Theorem 3.7 ([7, § 3]). *The functions τ and $\tilde{\tau}$ define the same divisor.*

Proof. At each step of the previous procedure, the coefficients $B_i^{(\nu)}(a)$ satisfy a Schlesinger system. Therefore, the form

$$\omega^{(\nu)} := \frac{1}{2}\sum_{i\neq j} \operatorname{tr}\left(B_i^{(\nu)}(a)B_j^{(\nu)}(a)\right)\frac{d(a_i - a_j)}{(a_i - a_j)}$$

is *closed*. Notice that $\omega^{(0)}$ is holomorphic on T (as we have $a_i \neq a_j$ if $i \neq j$), so that in particular $\omega^{(0)} = d\log h$ for h holomorphic and nonvanishing near a^*, but $\omega^{(\nu)}$ is only meromorphic for $\nu \geqslant 1$. More precisely we have:

Lemma 3.8 ([7, § 3]). *For any $\nu \geqslant 1$, we have $\omega^{(\nu)} - \omega^{(\nu-1)} = d\log\tau^{(\nu-1)}$, where $\tau^{(\nu-1)}$ is the equation obtained by the previous procedure when going from the step $\nu-1$ to the step ν.* □

At the final step ν_{final}, the form $\omega^{(\nu_{\text{final}})}$ is the form ω_{MJ} of Miwa and Jimbo for the original system (3.3). Setting $\tilde{\tau} = \prod_{\nu=0}^{\nu_{\text{final}}-1}\tau^{(\nu)}$, we find

$$\omega_{\text{MJ}} - \omega^{(0)} = d\log\tilde{\tau}.$$

As we know, by a theorem of Miwa, that ω_{MJ} represents Θ (in the sense of [11, Definition 6.1]), we obtain the equality $\operatorname{Div}\tau = \operatorname{Div}\tilde{\tau}$. □

Remark 3.9 (effectivity). Although it is in general difficult to compute the functions $B_{i,\ell m}^{(0)}$, and then the functions $\tau^{(\nu)}$, hence the function $\tilde{\tau}$, it is possible, in some examples, to compute the k-jets of these functions for k large enough, and to get information on the geometry of Θ as well as on the order of poles of the solutions of the Schlesinger system.

3.c The order of the pole along Θ of the solutions of the Schlesinger system

We start again with the situation of § 3.a with a system (3.2). Assume now that the size of the matrices A_i^o is 2 (*i.e.*, $d = 2$ above). We have initial data $(A_i^o, a_i^o)_{i=1,\ldots,n}$, and an isomonodromy deformation (3.3) of Schlesinger type (*i.e.*, the matrices A_i satisfy (Schl)) with a corresponding polar divisor Θ for the matrices A_i.

Make moreover the following assumptions:

(1) The monodromy representation defined by ∇^o on E^o is *irreducible*;

(2) at a point a^* of Θ, the splitting type of E_{a^*} is $(1, -1)$.

Remark 3.10. As E_{a^*} has degree 0 and is not trivial, its splitting type is $(k, -k)$ with $k \geqslant 1$. As the monodromy representation is irreducible and as $d = 2$, one has the bound $2k \leqslant n - 2$. When $n = 4$, Assumption (2) is therefore implied by Assumption (1).

Theorem 3.11 ([4, Theorem 2]). *Under these assumptions, for $i = 1, \dots, n$, the matrix $A_i(a)$ has a pole of order $\leqslant 2$ on Θ near a^*.*

We assume for convenience that none of the numbers a_i^* ($i = 1, \dots, n$) is 0 or ∞. We denote now by T a neighbourhood of a^*.

By the same[2] procedure as in § 3.b, introduce an apparent singularity (now at $x = 0$, not at $x = \infty$) to get a trivial bundle with connection E'_{a^*}.

The procedure described in § 3.b has only one step, because of Assumption (2).

The matrix of ∇ in a basis of E'_{a^*} is written as

$$B'(x)\,dx = \Big(\frac{B'_0}{x} + \sum_{i=1}^{n} \frac{B'_i}{x - a_i}\Big)\,dx$$

with $B'_0 = \operatorname{diag}(1, -1)$ and $\sum_{i=0}^{n} B'_i = 0$. Hence the entry $b'_{12}(x)$ of $B'(x)$ is holomorphic and vanishes at $x = 0$.

Lemma 3.12 ([4], p. 68). *Under Assumption* (1)*, the valuation* (*order of vanishing*) *of $b'_{12}(x)$ at $x = 0$ is $< n - 1$.*

Proof. Indeed, the coefficient of x^m ($m \geqslant 1$) in $b'_{12}(x)$ is $-\sum_{i=1}^{n} b'_{i,12}/a_i^{*(m-1)}$. If the valuation of $b'_{12}(x)$ is $\geqslant n - 1$, this implies that all $b'_{i,12}$ are zero, and (E'_{a^*}, ∇) is reducible, hence its monodromy too, in contradiction with Assumption (1). □

Applying the procedure (with one step) described in § 3.b, Andrey computes the equation of $\widetilde{\Theta}$ and finds that $\widetilde{\Theta}$ is smooth at a^*. This clearly implies that $\Theta = \widetilde{\Theta}$.

On the other hand, the original system (3.3) can be obtained by a simple base change from the system obtained after the previous procedure. A detailed computation shows that the original matrices $A_i(a)$ have a pole of order $\leqslant 2$ along Θ.

Remarks 3.13. (1) In [4], there is an explicit example where the order of the pole is 2.

(2) In [6, § 3], Andrey indicates that, without Assertion (2), a result similar to Theorem 3.11 still holds, but the order of the pole is $\leqslant 2k$, if E_{a^*} has splitting type $(k, -k)$.

4 Isomonodromic confluences

It is well known that a family of linear differential equations of one variable having only regular singularities may, for some values of the parameter, acquire an irregular singularity when various singular points for the generic value of the parameter merge

[2] However, Theorem 3.11 appeared in a Nice preprint dated July 1995, and the results of § 3.b were obtained later.

together. In the algebraic or analytic setting, we have a vector bundle E on $\overline{X}$ as in § 1 and we moreover assume that

- $\pi : \overline{X} \to T$ is smooth of relative dimension one,
- $\pi : Y \to T$ is finite (but Y is not necessarily smooth).

Given a relative connection $\nabla_{\overline{X}/T}$ on E, such that a generic fibre ∇_t on E_t has regular singularities on Y_t, it may happen that a special fibre ∇_{t^o} has an irregular singularity at some point of Y_{t^o}.

Exemple 4.1. Let $\overline{X} = \mathbb{C} \times \mathbb{C}$ with coordinates (x, t) and $Y = \{x^2 - t^2 = 0\}$. Take the trivial rank one bundle on $\overline{X}$ with the relative connection having the matrix $dx/(x^2 - t^2)$. For $t \neq 0$, we have a regular singularity at $x = \pm t$, and, for $t = 0$, we have an irregular singularity at $x = 0$.

The results below say that, under an isomonodromy condition, such a phenomenon does not appear.

A setting more general than the previous one happens to be useful. This occurs for instance when one considers confluence in a Schlesinger family parametrized by the universal covering of $(\mathbb{P}^1)^n \smallsetminus$ diagonals: the confluence takes place in the inverse image in T of a neighbourhood of the diagonals. Near a generic point of the diagonals, when only two points coincide, such an open set looks like the product of an upper half plane by an open set in $\mathbb{C}^{n-1}$, and one studies the confluence in vertical strips in this upper half plane.

4.a The algebraic/analytic case

This case was studied by Deligne [10] (see also [12]). Deligne used the full strength of Hironaka's theorem on resolution of singularities. His approach has been much simplified by Mebkhout, who gives a proof using resolution of singularities in dimension two only (*cf.* [13], [14], see also [15]).

We put here $\overline{X} = D \times T$, where D is a disc in $\mathbb{C}$. Let Y be a divisor in $\overline{X}$ on which the projection $\pi : \overline{X} \to T$ is finite. Let E be a holomorphic vector bundle on $\overline{X}$ equipped with a meromorphic *integrable* connection $\nabla : E \to \Omega^1_{\overline{X}}(*Y) \otimes_{\mathcal{O}_{\overline{X}}} E$ with poles along Y at most. It defines a meromorphic connection ∇_t on E_t with poles at the finite set of points Y_t for any $t \in T$.

Theorem 4.2 (Deligne). *Assume that, for generic $t \in T$, the connection ∇_t has only regular singularities at Y_t. Then this holds for any $t \in T$.*

Sketch of proof. The proof uses a variant of the Riemann Existence Theorem. One constructs a meromorphic bundle with a connection having regular singularities along Y at most, and having the same monodromy as the original connection. This auxiliary system satisfies the property of the theorem. Once such a system is constructed, one proves that it is isomorphic to the original one: one has an isomorphism between the two bundles with connection out of Y; due to the generic regular singularity of the original system, such an isomorphism is generically meromorphic along Y; by Hartogs, it is meromorphic. □

Moreover, it is then clear by a topological argument that, for $t^o \in T$, the monodromy of ∇_{t^o} on E_{t^o} around some point in Y_{t^o} can be computed as the product of well-chosen representatives of the monodromy operators of ∇_t on E_t near those points in Y_t which tend to the chosen point in Y_{t^o} when $t \to 0$.

4.b Other confluences

Denote by D an open disc centered at 0 in the complex plane and by Δ the open disc $\{|t-1|<1\}$. Denote by Y the intersection of $\overline{X} = D \times \Delta$ with the lines $x = a_i^o t$, with $a_i^o \in D$ for $i = 1, \dots, n$.

Consider an integrable (or isomonodromic) meromorphic system of differential equations on $\overline{X}$ with poles on Y, with matrix $\Omega = A(x,t)dx + C(x,t)dt$ having possible poles along Y only. Assume that the limits of $A(x,t)$, $C(x,t)$ when $t \to 0$ exist and are meromorphic on D with pole at 0 at most.

Theorem 4.3 ([3, Theorem 4]). *Assume that, for generic $t \in \Delta$, the system with matrix $A(x,t)dx$ has regular singularities at the points $a_i^o t$ $(i = 1, \dots, n)$. Then the limiting system at $t = 0$ has a regular singularity at $x = 0$.*

In such a situation, Theorem 4.2 does not apply. The method of proof given by Andrey is nevertheless similar: to construct a system with the same monodromy satisfying the property of the theorem, and then show, by an argument using Hartogs, that this system is isomorphic to the original one. The existence result uses the particular form of the polar divisor Y, by solving explicitly the Schlesinger system. We only give details for the existence part.

Proof. For simplicity, let us first consider, as in *loc. cit.*, the case where the monodromy around the boundary of D (*i.e.*, the product of well-chosen representatives of the monodromies around each a_i^o) is equal to the identity.

For the value $t = 1$ extend the system as a system on $\mathbb{C}$. Choose a point a_0^o distinct from the other a_i^o. There exists therefore, according to Plemelj, a Fuchsian system on $\mathbb{P}^1$ with no singularity at ∞ and an apparent singularity at a_0^o, having the monodromy of the original system. Let us write the matrix of this Fuchsian system as

$$\sum_{i=0}^{n} B_i \frac{dx}{x - a_i^o}, \quad \text{with } \sum_i B_i = 0.$$

The Schlesinger system (Schl) with respect to the parameter t describing the isomonodromic deformation with pole on $\widetilde{Y} = Y \cup \{x = a_0^o t\}$ can be written as

$$dB_i(t) = \sum_{j \neq i} [B_i(t), B_j(t)] \frac{dt}{t}$$

and therefore $\sum_i B_i(t)$ is constant, hence 0. The system can then be written as

$$dB_i(t) = \big[B_i(t), \textstyle\sum_j B_j(t)\big] \frac{dt}{t} = 0,$$

that is, the $B_i(t)$ are constant. The matrix of the connection (3.3) is written as $\sum_{i=0}^n B_i \, d(x - a_i^o t)/(x - a_i^o t)$, its limit when $t \to 0$ does exist and is equal to 0, hence we get the regular singularity of the system restricted at $t = 0$. The original system is then shown to be meromorphically isomorphic to the previous model[3].

If the monodromy around the boundary of D is not the identity, the previous construction can still be done, but we now have $\sum_{i=0}^n B_i = -B_\infty \neq 0$. In the Schlesinger system, we still have B_∞ constant, and $B_i(t)$ is a solution of the Fuchsian linear system

$$\frac{dB_i(t)}{dt} = \frac{\operatorname{ad} B_\infty}{t} \cdot B_i(t),$$

hence $B_i(t) = t^{B_\infty} B_i t^{-B_\infty}$. The matrix of the connection, which is as in (3.3):

$$\Omega = \sum_{i=0}^n B_i(t) \frac{d(x - a_i^o t)}{(x - a_i^o t)}$$

satisfies, out of $x = 0$,

$$\lim_{t \to 0} x\Omega = B_\infty \, dx,$$

hence we get the regular singularity at the limit. □

4.c Confluence as a dynamical version of apparent singularities

Let us begin with preliminary remarks. Let a^* be a set of n distinct points in $\mathbb{P}^1$ and let E_{a^*} be a *nontrivial* holomorphic bundle of degree 0 with a connection ∇_{a^*} having logarithmic poles at a^*. If T is as in § 3.a, we have, after Theorem 3.1 applied to the initial condition (E_{a^*}, ∇_{a^*}), a vector bundle E on $\mathbb{P}^1 \times T$ with a logarithmic connection having poles on Y, which restricts to the initial condition at $a = a^*$.

The bundle with connection (E_{a^*}, ∇_{a^*}) is contained in a meromorphic bundle with connection $(E_{a^*}(*a^*), \nabla_{a^*})$, in which the Riemann–Hilbert problem may or may not have a solution.

[3] Some eigenvalues of some B_i may differ by a nonzero integer: this usually happens at the apparent singularity; this would not happen in Deligne's method where the choice of a "Deligne extension" allows one, by its uniqueness, to construct a Fuchsian system in a global situation (after resolution of singularities) by a local procedure.

One can ask the question: *Is E_a trivial for generic a?*

Andrey has given examples of a monodromy representation for which the Riemann–Hilbert problem has no solution, whatever the choice of a position a of the poles could be (*cf.* [1, Proposition 5.2.1, p. 126]). In particular, the answer to the previous question may be negative.

On the other hand, if one allows confluence, he has obtained the following result in [5].

Let E'_{a^*} be a trivial holomorphic subbundle of $E_{a^*}(*a_n)$ on which the connection has a pole of Poincaré rank $r \geqslant 1$ at a_n. It is then possible to construct an isomonodromic confluence (as in § 4.b) of trivial bundles E'_t with logarithmic connection ∇_t (*i.e.*, Fuchsian systems) having poles at $a_1, \dots, a_n$ and at a finite number of distinct points $b_j(t)$ which converge to a_n when $t \to 0$, so that the limit bundle is E'_{a^*}. The points $b_j(t)$ are apparent singularities for ∇_t and their number can be bounded by $(rd(d-1)/2)^2$, using a result of E. Corel [9].

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Two notions of integrability

Michèle Audin

Institut de Recherche Mathématique Avancée
Université Louis Pasteur et CNRS
7 rue René Descartes
67084 Strasbourg cedex, France
email: `maudin@math.u-strasbg.fr`

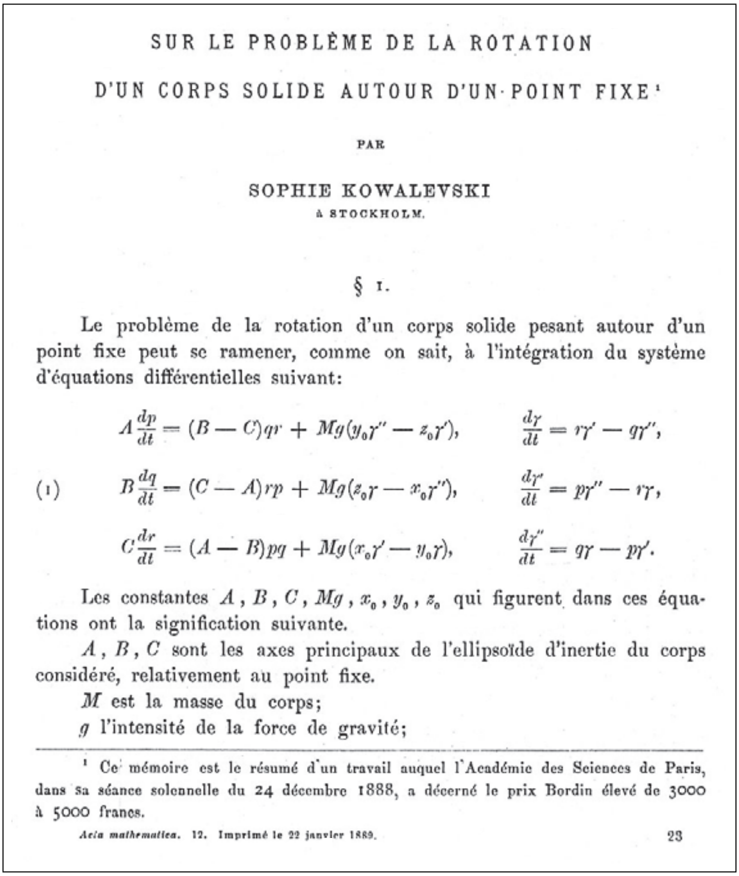
Je me souviens qu'Andrei
admirait Kowalevskaya

Abstract. We investigate the relation between two notions of integrability, to have enough first integrals on the one hand, and to have meromorphic solutions on the other, that are present in Kowalevskaya's famous mémoire on the rigid body problem. We concentrate on the examples of the rigid body and of the system of Hénon–Heiles.

Contents

I remember Andrei sitting in my office in Strasbourg and telling me his admiration for Sofia Kowalevskaya. I am not sure whether this was because I am a woman, or because he knew I had written a long paper on the "Kowalevski top" [7] or simply because he was very enthusiastic over her brilliant and romantic personality. I would have been very happy to discuss the topic of the present paper with him.

It is with deep sadness, thinking both of Sofia and Andrei, that I reproduce (in Figure 1), the very first page of the beautiful paper [13] on the rigid body. This paper is rather famous. This is the work for which her author won the Bordin prize of the French Academy of Sciences in 1888. It seems to me that this is still very interesting and modern, in particular because of the two notions of integrability that appear here. I hope that the present paper will participate in showing how seminal and important is Kowalevskaya's contribution to the theory of integrable systems (see e.g. [8] for other aspects of her work).



SUR LE PROBLÈME DE LA ROTATION

D'UN CORPS SOLIDE AUTOUR D'UN POINT FIXE[1]

PAR

SOPHIE KOWALEVSKI

À STOCKHOLM.

§ 1.

Le problème de la rotation d'un corps solide pesant autour d'un point fixe peut se ramener, comme on sait, à l'intégration du système d'équations différentielles suivant:

$$(1)\qquad \begin{aligned} A\frac{dp}{dt} &= (B-C)qr + Mg(y_0\gamma'' - z_0\gamma'), & \frac{d\gamma}{dt} &= r\gamma' - q\gamma'', \\ B\frac{dq}{dt} &= (C-A)rp + Mg(z_0\gamma - x_0\gamma''), & \frac{d\gamma'}{dt} &= p\gamma'' - r\gamma, \\ C\frac{dr}{dt} &= (A-B)pq + Mg(x_0\gamma' - y_0\gamma), & \frac{d\gamma''}{dt} &= q\gamma - p\gamma'. \end{aligned}$$

Les constantes A, B, C, Mg, x_0, y_0, z_0 qui figurent dans ces équations ont la signification suivante.

A, B, C sont les axes principaux de l'ellipsoïde d'inertie du corps considéré, relativement au point fixe.

M est la masse du corps;

g l'intensité de la force de gravité;

[1] Ce mémoire est le résumé d'un travail auquel l'Académie des Sciences de Paris, dans sa séance solennelle du 24 décembre 1888, a décerné le prix Bordin élevé de 3000 à 5000 francs.

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Figure 1. The first page of [13].

I will concentrate on a very small part of the mathematics contained in [13] and limit myself to a discussion of these two notions of integrability and to try to understand their relations. Let me first explain what these two notions are. Here I will only deal with complex analytic Hamiltonian systems. The two properties are the following:

(K) The singularities of the solutions are poles (here (K) is for Kowalevskaya[1]).

(L) There are enough first integrals (here (L) is for Liouville).

They are not logically related in the sense that (as far as I know) none implies the other. However, in many significant examples, one is satisfied if and only if the other one is. Moreover, it may happen that some systems are suspected *not* to be integrable (in the Liouville sense) because they do not satisfy (K); this is the case for instance of the Hénon–Heiles system (see the discussion in [17]).

To make things more tractable, I will use "soft" versions of the two properties (K) and (L), replacing the differential (Hamiltonian) system by a *linear* differential system, the variational equation along a previously chosen particular solution:

(H) The monodromy around the singularities is trivial (here (H) is for Haine).

(MR) The Galois group is virtually Abelian (here (MR) is for Morales–Ramis).

I will discuss this more precisely (and give the appropriate definitions and explanations) on a few examples. I will of course mainly focus on the examples studied in the seminal paper [13]. The motivations for looking at these examples and at this paper are numerous. Among them:

- This is the place where the relation between the two notions of integrability I want to discuss appears for the first time.
- The problem of the rigid body is a most classical problem and I believe that we should study classical problems.

Acknowledgements. I thank all the people who made comments on previous versions of this paper, especially those who attended the talks I gave on this work in Strasbourg and in Reykjavik, and the referee for his nice but useful comments.

I am deeply grateful to Andrzej Maciejewski for his careful reading and his hunting of misprints and more serious errors.

1 The rigid body

1.1 The differential system

In the first page of her paper [13], Sofia Kowalevskaya writes the differential system describing the motion of a rigid body with a fixed point in a constant gravitation field. The equations are written in a frame which is attached to the body, the relative frame.

[1] Notice that this property is often called the "Kowalevski–Painlevé property" by the people working on integrable systems, probably in reference with the "Painlevé property" which is, for a differential equation, the fact that its *mobile* singularities are poles.

$$
\begin{aligned}
A\frac{dp}{dt} &= (B-C)qr + Mg(y_0\gamma'' - z_0\gamma'), & \frac{d\gamma}{dt} &= r\gamma' - q\gamma'',\\
B\frac{dq}{dt} &= (C-A)rp + Mg(z_0\gamma - x_0\gamma''), & \frac{d\gamma'}{dt} &= p\gamma'' - r\gamma,\\
C\frac{dr}{dt} &= (A-B)pq + Mg(x_0\gamma' - y_0\gamma), & \frac{d\gamma''}{dt} &= q\gamma - p\gamma'.
\end{aligned}
$$

Figure 2. The differential system.

The quantities M and g which appear in these equations are the mass of the body and a gravitational constant; we shall choose the units so that $Mg = 1$. The vector

$$\Gamma = \begin{pmatrix}\gamma\\ \gamma'\\ \gamma''\end{pmatrix}, \text{ that I prefer to write } \begin{pmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\end{pmatrix}$$

is the gravity field. Let me call M the total angular momentum and Ω the angular velocity, two vectors that are related by $M = \mathcal{J}\Omega$, for a symmetric definite positive matrix $\mathcal{J}$, the inertia matrix (which reflects the shape of the body). Here[2],

$$\Omega = \begin{pmatrix}p\\ q\\ r\end{pmatrix}, \text{ and } \mathcal{J} = \begin{pmatrix}A & 0 & 0\\ 0 & B & 0\\ 0 & 0 & C\end{pmatrix}, \text{ so that } M = \begin{pmatrix}Ap\\ Bq\\ Cr\end{pmatrix}.$$

Notice (see [2, p. 141]) that A, B and C must satisfy the triangle inequalities

$$A + B \geq C, \quad B + C \geq A, \quad C + A \geq B$$

(equality holding if and only if the body is planar). The center of gravity G and the fixed point O are related by the constant vector

$$\overrightarrow{OG} = L = \begin{pmatrix}x_0\\ y_0\\ z_0\end{pmatrix}.$$

Denoting with a dot the derivation with respect to time, our differential system can also be written in a more compact form:

$$\dot{M} = M \times \Omega + L \times \Gamma, \quad \dot{\Gamma} = \Gamma \times \Omega.$$

[2] The symmetric matrix $\mathcal{J}$ is diagonalizable, so that we will assume that it is diagonal. Notice that the fact that the symmetric real matrices are diagonalizable was proven by Lagrange [14]… because he needed it to deal with the inertia matrix of the spinning top.

1.2 What was known before Kowalevskaya's paper

Let us now turn the page. Sofia Kowalevskaya reminds us (Figure 3) that, at that time, *jusqu'à présent*[3], it has been possible to solve these equations only in two special cases:

> Jusqu'à présent on n'était parvenu à intégrer ces équations que dans deux cas particuliers:
>
> 1) Le cas de POISSON (ou d'EULER) où l'on a $x_0 = y_0 = z_0 = 0$,
>
> 2) Le cas de LAGRANGE où l'on a $A = B$, $x_0 = y_0 = 0$.
>
> Dans ces deux cas l'intégration s'opère à l'aide des fonctions $\vartheta(u)$ dont l'argument est une fonction entière linéaire du temps.
>
> Les six quantités $p, q, r, \gamma, \gamma', \gamma''$ sont dans ces deux cas des fonctions uniformes du temps, n'ayant d'autres singularités que des pôles pour toutes les valeurs finies de la variable.

Figure 3

- the Euler–Poisson case, where $O = G$ (the center of gravity is the fixed point);
- the Lagrange case, in which the body rotates about an axis of revolution (the vector $\overrightarrow{OG}$ is the third vector of an orthonormal basis in which the constants (A, B, C) have the form (A, A, C)).

She notices that, in these two cases, the solutions can be written in terms of ϑ-functions. In (slightly more) geometric terms, this means that there is an elliptic curve present and that the solutions are linear on this curve. More importantly, she writes "the six quantities $p, q, r, \gamma, \gamma', \gamma''$ are in these two cases uniform functions of time, having no other singularities than poles for all finite values of the variable". And (Figure 4)

> Les intégrales des équations différentielles considérées conservent-elles cette propriété dans le cas général?

Figure 4

"Do the solutions of the differential equations still have this property in the general case?"

Remarks. (1) Notice that the differential system under consideration is non linear. Its solutions can thus have very complicated singularities (essential singularities, branching, logarithms,...) that may depend on the initial conditions and not only on the coefficients.

[3] Her French was much better than the so called English used nowadays in the mathematical papers (in the present one, for instance). This was the language she used, for instance, in her correspondence with Mittag-Leffler (see [8]).

(2) Moreover, if one considers time as a real variable (which is what people were doing until Kowalevskaya), there are no singularities at all: the energy is a proper function, so that nothing goes to infinity, the motion takes place on compact manifolds.

(3) This is the reason why there was no progress on the problem during about one century: to do something new, it was necessary to use complex analysis. And Sofia Kowalevskaya was one of the best specialists of this (new) topic at that time.

Then she computes and concludes that "this property", namely what we call property (K), is satisfied only in the two cases mentioned above... and in a new case, which is now called the "Kowalevski top", this is the case where

$$A = B = 2C, \quad z_0 = 0,$$

there is an axis of revolution but this is *not* the axis $\overrightarrow{OG}$, as this was the case for the Lagrange top, the axis is rather orthogonal to $\overrightarrow{OG}$.

1.3 The property (L)

Now comes the best. Let me explain this in modern geometric terms. The differential system is Hamiltonian. To avoid technicalities and the introduction of Poisson structures, let me just concentrate on the submanifold

$$W_{2\ell} = \left\{(\Gamma, M) \in \mathbb{C}^6 \mid \|\Gamma\|^2 = 1 \text{ and } \Gamma \cdot M = 2\ell\right\}.$$

This is a 4-dimensional manifold. The two equations correspond to the obvious fact that the intensity $\|\Gamma\|$ of the gravitation field is fixed (a consequence of the fact that Γ is constant in the absolute frame) and that $\Gamma \cdot M$ must also be conserved in application of the "law of areas". The manifold $W_{2\ell}$ carries a natural symplectic structure (see for instance [3]), we can consider the total energy[4]

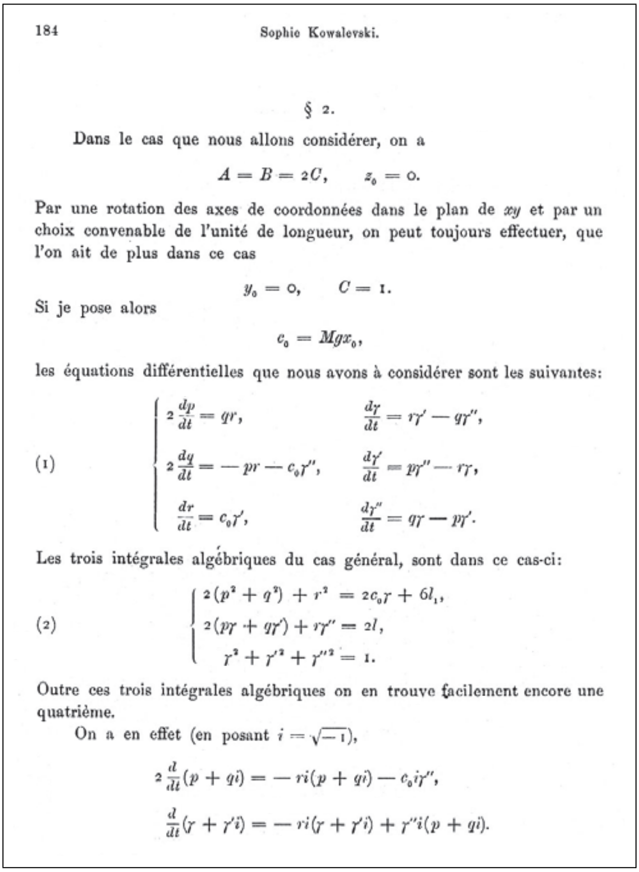
$$H(\Gamma, M) = \frac{1}{2} M \cdot \Omega - \Gamma \cdot L$$

as a function on $W_{2\ell}$ and the differential system under consideration is the corresponding Hamiltonian system. So that the total energy is also a conserved quantity (this is called $3\ell_1$ in Kowalevskaya's paper, a notation visible on Figure 5).

Notice that we have a Hamiltonian system on a 4-dimensional manifold. And remember that what Kowalevskaya wants to do is *to solve* the equations. So that she notices, incidentally, that, "in addition to these three algebraic integrals, it is easy to find a fourth one". What she was interested in was to *solve* the equations. Then she uses these four conserved quantities to actually solve the system in her case. This takes the rest of the paper. She writes the solutions, explicitly, in terms of ϑ-functions associated with a genus-2 curve.

Of course, what she notices is that the Hamiltonian system is what we call nowadays "Liouville integrable" in her case[5]. We have a Hamiltonian system with two degrees

[4] We adopt here the sign convention used in [13].


184 Sophie Kowalevski.

§ 2.

Dans le cas que nous allons considérer, on a

$$A = B = 2C, \qquad z_0 = 0.$$

Par une rotation des axes de coordonnées dans le plan de xy et par un choix convenable de l'unité de longueur, on peut toujours effectuer, que l'on ait de plus dans ce cas

$$y_0 = 0, \qquad C = 1.$$

Si je pose alors

$$c_0 = Mgx_0,$$

les équations différentielles que nous avons à considérer sont les suivantes:

$$(1) \qquad \begin{cases} 2\dfrac{dp}{dt} = qr, & \dfrac{d\gamma}{dt} = r\gamma' - q\gamma'', \\ 2\dfrac{dq}{dt} = -pr - c_0\gamma'', & \dfrac{d\gamma'}{dt} = p\gamma'' - r\gamma, \\ \dfrac{dr}{dt} = c_0\gamma', & \dfrac{d\gamma''}{dt} = q\gamma - p\gamma'. \end{cases}$$

Les trois intégrales algébriques du cas général, sont dans ce cas-ci:

$$(2) \qquad \begin{cases} 2(p^2 + q^2) + r^2 = 2c_0\gamma + 6l_1, \\ 2(p\gamma + q\gamma') + r\gamma'' = 2l, \\ \gamma^2 + \gamma'^2 + \gamma''^2 = 1. \end{cases}$$

Outre ces trois intégrales algébriques on en trouve facilement encore une quatrième.

On a en effet (en posant $i = \sqrt{-1}$),

$$2\frac{d}{dt}(p + qi) = -ri(p + qi) - c_0 i\gamma'',$$

$$\frac{d}{dt}(\gamma + \gamma' i) = -ri(\gamma + \gamma' i) + \gamma'' i(p + qi).$$

Figure 5. Liouville integrability in [13].

of freedom (that is, on a 4-dimensional symplectic manifold), so that the Kowalevski integral

$$K = \left|(p + iq)^2 + (\gamma_1 + i\gamma_2)\right|^2$$

makes it an integrable system in the Liouville sense. She does not mention this property of "Liouville integrability" as such, but of course she uses the first integrals to solve the equations.

Let me add that the two special cases already solved before [13] are of course Liouville integrable, since

- in the Euler–Poisson case, $K = \|M\|^2$ is a second first integral;
- in the Lagrange case, the momentum with respect to the axis of revolution, $K = M \cdot L$, is obviously a conserved quantity.

[5] The readers can look at [4] for instance.

1.4 What is integrability?

What Sofia Kowalevskaya showed in her paper is that, for the rigid body problem, if Property (K) is satisfied, then we are in cases where Property (L) is also satisfied (and this allows to integrate the Hamiltonian system). Hence, (K) implies (L) here.

This raises a few questions (Notice that the title of this paragraph is borrowed to [19], a book where the questions raised here are investigated):

(1) Does (K) implies (L) more generally?

(2) Does (L) implies (K) for this system?

(3) Does (L) implies (K) in general?

(4) In which cases of the rigid body is (L) satisfied?

Let us look now at the answers we know to these questions in the case of the rigid body.

Theorem 1.1 (Husson, Ziglin, Maciejewski and Przybylska). *If the rigid body is integrable in the Liouville sense, then*

- *either we are in the Euler, Lagrange or Kowalevskaya case,*
- *or* $A = B = 4C$, $z_0 = 0$ *and there is an additional first integral on* W_0.

The additional case is the Goryachev–Chaplygin case, where, only on the manifold W_0 (that is, for $2\ell = 0$),

$$K = r(p^2 + q^2) + p\gamma_3$$

is a first integral. The fact that the three and half mentioned cases are the only ones in which the system has an *algebraic* additional integral has been proved by Husson [10]. With a *meromorphic* first integral, it has been proved by Ziglin [21] using techniques which were invented by him [20] and are close to the ones I want to discuss here, namely properties of the monodromy groups of the (linearized) differential system. An alternative "Galoisian" proof has been given recently, using the Morales–Ramis criterion [17], [18], by Maciejewski and Przybylska [16].

I will come back to the Goryachev–Chaplygin case later. Notice that Kowalevskaya does not find this case with her analysis and indeed, there are non meromorphic solutions in this case. This is a Liouville integrable case that does not satisfy the Kowalevskaya condition. Note however that, for the rigid body (K) $\Rightarrow$ (L).

2 Integrability (up to order 1), two criteria

Let us now linearize the problem. We consider a complex analytic Hamiltonian system, that is, an analytic vector field X_H on an open subset of an affine space $\mathbb{C}^m$ which is the Hamiltonian vector field for some function H defined on symplectic submanifolds of $\mathbb{C}^m$.

2.1 The variational equation

We consider a special solution of the (nonlinear in general) differential system

$$\dot{x}(t) = X_H(x(t)).$$

This is a map

$$\begin{aligned} U &\longrightarrow \mathbb{C}^m, \\ t &\longmapsto \varphi^t(x_0), \end{aligned}$$

where U is a Riemann surface (a quotient of an open subset of $\mathbb{C}$), φ^t is the flow of X_H and we are describing here the special solution passing through a given point x_0.

I shall of course use the example of the rigid body, but let me add here a simple (and academic) example.

Example, a Hénon–Heiles system. In this example, the symplectic manifold is the space $\mathbb{C}^4$ itself, endowed with the symplectic form

$$\omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$$

and the Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}Aq_1^2 - q_1^2 q_2.$$

The Hamiltonian system is

$$\begin{cases} \dot{q}_1 = p_1, \\ \dot{q}_2 = p_2, \\ \dot{p}_1 = -Aq_1 + 2q_1q_2, \\ \dot{p}_2 = q_1^2. \end{cases}$$

Here are two special solutions of this system:

- the Riemann surface is $\mathbb{C} - \{0\}$ and

$$\begin{cases} q_1 = \dfrac{3\sqrt{2}}{t^2}, & p_1 = -\dfrac{6\sqrt{2}}{t^3}, \\ q_2 = \dfrac{3}{t^2} + \dfrac{A}{2}, & p_2 = -\dfrac{6}{t^3}; \end{cases}$$

- the Riemann surface is $\mathbb{C}$ and

$$\begin{cases} q_1 = 0, & p_1 = 0, \\ q_2 = at + b, & p_2 = a \end{cases}$$

  for some constants a and b.

Then, coming back to the general framework, we can linearize the system along the given solution. This can be defined intrinsically (see, for instance [4, § III.1]), but here we are in $\mathbb{C}^m$, so that the Hamiltonian vector field X_H itself can be considered as a map $\mathbb{C}^m \to \mathbb{C}^m$ and can be differentiated. The variational equation along the solution $x(t)$ is simply the linear differential system

$$\dot{y} = (dX_H)_{x(t)} \cdot y.$$

Example, Hénon–Heiles, continuation. Along the solutions given above, using capital letters to denote the variations of the low case letters, the linear system is

$$\begin{cases} \dot{Q}_1 = P_1, & \dot{P}_1 = \dfrac{6}{t^2} Q_1 + \dfrac{6\sqrt{2}}{t^2} Q_2, \\ \dot{Q}_2 = P_2, & \dot{P}_2 = \dfrac{6\sqrt{2}}{t^2} Q_1 \end{cases}$$

for the first special solution and

$$\begin{cases} \dot{Q}_1 = P_1, & \dot{P}_1 = (2at + b - A) Q_1, \\ \dot{Q}_2 = P_2, & \dot{P}_2 = 0 \end{cases}$$

for the second one.

2.2 Haine's criterion for (K)

In a beautiful paper [9] devoted to the investigation of geodesic flows on $\mathrm{SO}(n)$, Haine noticed the following simple and useful property:

Theorem 2.1 (Haine [9]). *If a complex analytic Hamiltonian system on $\mathbb{C}^m$ satisfies the Kowalevski property* (K)*, then the monodromy around the poles of the solutions of the variational equation along any solution is trivial.*

Let us illustrate this theorem by our simple example.

Example, Hénon–Heiles, continuation. This is really an academic example and it was already used in [1]. We consider the special solution above with a pole at 0 (as in Example 2.1) and look at the variational equation given in § 2.1. Changing of unknown functions

$$Q_1 = t^{-2} x_1, \quad Q_2 = t^{-2} x_2, \quad P_1 = t^{-3} y_1, \quad P_2 = t^{-3} y_2$$

gives the new system

$$\frac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} = \frac{1}{t} \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 6 & 6\sqrt{2} & 3 & 0 \\ 6\sqrt{2} & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix}.$$

In order that the solutions be univalued, it is necessary that the differences between the eigenvalues of the matrix be integral. But its characteristic polynomial is

$$\lambda^4 - 10\lambda^3 + 31\lambda^2 - 30\lambda - 72 = (\lambda + 1)(\lambda - 6)(\lambda^2 - 5\lambda + 12),$$

so that the property is obviously not satisfied and our system does not satisfy (K).

Remark. From the point of view of Sofia Kowalevskaya's paper, this system is not fully satisfactory: it is very easy to check that the special solutions I have written (in Example 2.1) are the only ones that have no other singularities that poles. Notice that the two families are very different… and, by the way, that there are no constants of integration in the solution with a pole at 0. In her paper, Kowalevskaya says (Figure 6) that the series giving the solutions in the case of the rigid body should

> séries, pour pouvoir représenter le système général d'intégrales des équations différentielles considérées, devraient contenir *cinq* constantes arbitraires.

Figure 6

contain *five* arbitrary constants. This is what is called a "principal balance" (see the papers in [19]). There are not enough constants of integration in our example: the Hénon–Heiles system has no principal balance. There are not enough meromorphic solutions.

2.3 Morales and Ramis's criterion for (L)

This is a criterion of the same nature, since it also deals with the variational equation. See the original papers [17], [18] and also [4], [5].

Theorem 2.2 (Morales and Ramis [17], [18]). *If a complex analytic Hamiltonian system on $\mathbb{C}^m$ satisfies the Liouville integrability property* (L)*, then the Galois group of the variational equation along any solution is virtually Abelian.*

Recall that to say that an algebraic group is *virtually* Abelian is to say that its neutral component is an Abelian group.

Example, Hénon–Heiles, still more academic. For instance, the variational equation along the second solution exhibited in the Hénon–Heiles examples in § 2.1 reduces to

$$\ddot{Q}_1 = (2at + 2b - A)Q_1,$$

an Airy equation. If we accept to compactify the Riemann surface $\mathbb{C}$ which is our special solution as a complex projective line, our ground field is $\mathbb{C}(t)$ and the Galois group of the Airy equation is $SL(2;\mathbb{C})$ (see [12]), a non Abelian connected group. Hence the Hénon–Heiles system is not Liouville integrable (at least with a rational first integral).

I like this example very much, because this is really a beautiful academic example. Firstly, I have given two completely different arguments for the seemingly different properties (K) and (L). Secondly, it shows that the Galois group is something very rich. Of course, it contains the monodromy group of the variational equation, and hence also its Zariski closure. But, in this simple example, the Riemann surface is simply connected so that there is no monodromy at all. Which does not prevent the Galois group of being huge.

Recall however that, if the variational equation has only regular singularities (which is not the case at infinity in our example), the Galois group contains nothing more than the (closure of the) monodromy group.

2.4 What is integrability? A linear version

This allows us to translate our questions (in § 1.4) into questions relative to the variational equations, namely what is the relation between property (H) and property (MR)?

$$\begin{array}{ccc} \text{Kowaleski property} & \overset{?}{\longleftrightarrow} & \text{Liouville integrability} \\ \text{Haine}\ \Big\Downarrow & & \Big\Downarrow\ \text{Morales–Ramis} \\ \text{trivial monodromy around poles} & \underset{?}{\longleftrightarrow} & \text{virtual commutativity of Galois group} \end{array}$$

Having worked on quite a few examples of applications of the (MR) criterion, I was surprised that, very often, the particular solution used to test this criterion is an elliptic curve minus a certain number of points.

There is a practical reason: when the variational equation has only regular singular points, both criteria are dealing with monodromy groups. The fact that the monodromy about the poles is trivial implies that the monodromy group is Abelian (Figure 7). Hence, if (H) is satisfied, (MR) is.

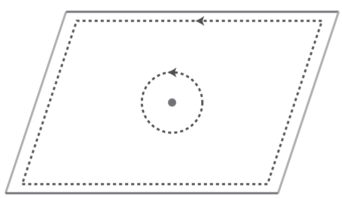


Figure 7

But there is probably a more serious reason. In general, we consider a family of Hamiltonians, H_a, say, depending on a parameter a. For some values of a, we know that the system is integrable. Integrable systems are often algebraically integrable, which means that the common levels of the first integrals are (open subsets of) Abelian varieties, in general coverings of hyperelliptic Jacobians. This is the complex algebraic counterpart of the Liouville tori in (real) symplectic topology. And, of course, the solutions of the Hamiltonian system stay on these Abelian varieties. One expects these varieties to degenerate to cases which contain elliptic curves[6].

3 Comparison of Haine's and Morales–Ramis's criteria

The case of geodesic flows of invariant metrics on the Lie group $SO(N)$ was investigated in [9]. This is related with the rigid body (the case $N = 3$ includes the Euler case). Haine shows that, the cases where (H) is satisfied are exactly those for which we know that the system is Liouville integrable.

The case of the rigid body

The case of the rigid body is slightly more involved, contrarily to what I was believing when I wrote a first version of this paper. I will only consider a special case of rigid body: the body has a rotational symmetry (an axis of revolution), which means that the inertia matrix has a double eigenvalue, and the "axis" $\overrightarrow{OG}$ lies in the "equatorial" plane, the eigenplane corresponding to the double eigenvalue.

Trying to compare (H) and (MR) in this case, I will prove:

Proposition 3.1. *Assume that in the Euler–Poisson equations, $A = B$ and $z_0 = 0$. If the solutions are meromorphic functions of time, then $A/C = 1$ or 2.*

And I will relate the proof to that of Ziglin's theorem (here Theorem 1.1) in this special case, namely:

Theorem 3.2. *Assume that in the Euler–Poisson equations, $A = B$ and $z_0 = 0$. If there exists an additional real meromorphic first integral, then $A/C = 1$, 2 or 4.*

Notice that

- the case $A/C = 1$ is a special case of a Lagrange top,
- the case $A/C = 2$ is the Kowalevski case,
- the case $A/C = 4$ is the Goryachev–Chaplygin case,

hence the three cases are known to be integrable (at least on W_0 for the last one).

[6] I am indebted to the referee for the clarification of this remark.

3.1 Choice of special solutions

We can assume that $x_0 = 1$ and $y_0 = z_0 = 0$ (by a change of coordinates) and that $C = 1$ (by a change of units).

Following Ziglin, we shall consider two families of solutions,

(1) with $p = r = \gamma_2 = 0$;

(2) or with $p = q = \gamma_3 = 0$.

Notice that such solutions will stay on the symplectic manifold W_0 (that is, $M \cdot \Gamma = 0$). The Hamiltonian system reduces to

(1) in the first case,

$$\begin{cases} A\dot{q} = -\gamma_3, \\ \dot{\gamma}_1 = -q\gamma_3, \\ \dot{\gamma}_3 = q\gamma_1; \end{cases}$$

(2) and in the second case

$$\begin{cases} \dot{r} = \gamma_2, \\ \dot{\gamma}_1 = r\gamma_2, \\ \dot{\gamma}_2 = -r\gamma_1. \end{cases}$$

The solutions are supported by

(1) the curve $\mathcal{E}_h$

$$\begin{cases} \gamma_1^2 + \gamma_3^2 = 1, \\ \frac{1}{2}Aq^2 - \gamma_1 = h; \end{cases}$$

(2) the curve $\mathcal{F}_h$

$$\begin{cases} \gamma_1^2 + \gamma_2^2 = 1, \\ \frac{1}{2}r^2 - \gamma_1 = h \end{cases}$$

(h is a parameter, the value of the total energy H on this solution). Our solution curves are thus intersections of two quadrics, hence in general elliptic curves. Figure 8 shows the shape of (the real part of) the curves $\mathcal{F}_h$ as h varies.

The picture for $\mathcal{E}_h$ is completely analogous. Two points at infinity, in the directions $(\gamma_1, \gamma_2, r) = (1, \pm i, 0)$ (or $(\gamma_1, \gamma_3, q) = (1, \pm i, 0)$) are missing on theses curves.

Remark. Notice that both $\mathcal{E}_h$ and $\mathcal{F}_h$ are smooth if and only if $h \neq \pm 1$. For $h = \pm 1$, they have an ordinary double point.

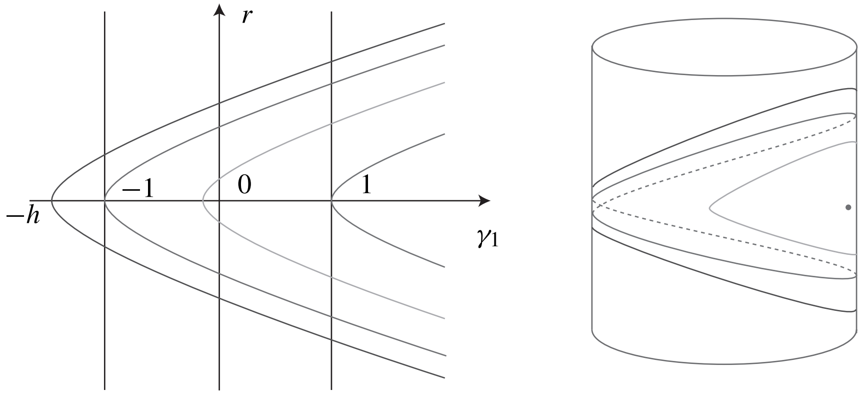


Figure 8. The curves $\mathcal{F}_h$.

3.2 The variational equation

Using as in the example above, a capital letter to denote the variations of the variable denoted by low case letters, the variational equation along a curve $\mathcal{E}_h$ is the linear differential system

$$\begin{cases} A\dot{P} = (A-1)qR, \\ A\dot{Q} = \Gamma_1, \\ \dot{R} = \Gamma_2, \end{cases} \qquad \begin{cases} \dot{\Gamma}_1 = -q\Gamma_3 - \gamma_3 Q, \\ \dot{\Gamma}_2 = \gamma_3 P - \gamma_1 R, \\ \dot{\Gamma}_3 = q\Gamma_1 + \gamma_1 Q. \end{cases}$$

Using the fact that the vector $(P, Q, R, \Gamma_1, \Gamma_2, \Gamma_3)$ is tangent to W_0, namely, the relation

$$A(\gamma_1 P + q\Gamma_2) + \gamma_3 R = 0,$$

the system reduces to the linear equation

$$\ddot{R} + \frac{q\gamma_3}{\gamma_1}\dot{R} + \left(\frac{\gamma_3^2}{A\gamma_1} + \gamma_1\right)R = 0.$$

In a similar way, along a solution of the family $\mathcal{F}_h$, we find

$$\begin{cases} A\dot{P} = (A-1)rQ, \\ A\dot{Q} = (1-A)rP - \Gamma_3, \\ \dot{R} = \Gamma_2, \end{cases} \qquad \begin{cases} \dot{\Gamma}_1 = r\Gamma_2 + \gamma_2 R, \\ \dot{\Gamma}_2 = -r\Gamma_1 - \gamma_1 R, \\ \dot{\Gamma}_3 = \gamma_1 Q - \gamma_2 P. \end{cases}$$

Differentiating the equation for $\dot{\Gamma}_3$ with respect to t and using the relations between our variables to simplify, we get

$$\begin{aligned}\ddot{\Gamma}_3 &= r(\gamma_1 P + \gamma_2 Q) + \gamma_1\left(\frac{1-A}{A} rP - \frac{\Gamma_3}{A}\right) - \gamma_2\left(\frac{A-1}{A} rQ\right)\\ &= \frac{1}{A}\left(r(\gamma_1 P + \gamma_2 Q) - \gamma_1 \Gamma_3\right)\\ &= -\frac{1}{A}\left(\frac{r^2}{A} + \gamma_1\right)\Gamma_3.\end{aligned}$$

So that we are reduced to the investigation of the linear differential equation of order 2

$$\ddot{\Gamma}_3 = -\frac{1}{A}\left(\frac{r^2}{A} + \gamma_1\right)\Gamma_3$$

where r and γ_1 are solutions of the non linear system above.

3.3 About the poles

Let us now look at the monodromy around the poles. About a point at infinity of the elliptic curve $\mathcal{E}_h$ (resp. $\mathcal{F}_h$), the solutions of the non linear system have the form

$$\begin{cases}\gamma_1 = -2At^{-2}(1 + tg_1(t)),\\ \gamma_3 = 2Ait^{-2}(1 + tg_3(t)),\\ r = 2it^{-1}(1 + tg_2(t))\end{cases} \quad \text{resp.} \quad \begin{cases}\gamma_1 = -2t^{-2}(1 + tf_1(t)),\\ \gamma_2 = -2it^{-2}(1 + tf_2(t)),\\ q = 2it^{-1}(1 + tf_3(t))\end{cases}$$

for some functions f_i's, g_i's, that are holomorphic at 0.

Remark. Notice that these functions are of course elliptic functions of the time t (they parametrize an elliptic curve), so that we could have written them "explicitly" in terms of the Jacobi functions sn, cn and even dn. I always find it very hard for a geometer to read such formulas. This might be less impressive to write the solutions the way I wrote them here, but since we will not need more, I will content myself with these formulas, that follow directly from the differential system.

Let us look now at the indicial equations for our two linear differential equations. For the first family of solutions (along the curve $\mathcal{E}_h$), this is

$$s^2 + s + 2(1 - A) = 0,$$

the difference of the two roots of which is $\sqrt{8A-7}$.

For the second differential equation (along the curve $\mathcal{F}_h$), this is

$$s^2 - s - \frac{2}{A}\left(\frac{2}{A} + 1\right) = \left(s + \frac{2}{A}\right)\left(s - \frac{2}{A} - 1\right) = 0.$$

In this case, the two roots are $-2/A$ and $2/A + 1$.

For the monodromy around the two points at infinity of our elliptic curve $\mathcal{F}_h$ to be trivial, it is necessary that the roots of the indicial equations be integers, namely here that $2/A \in \mathbf{Z}$. Recall the triangle inequalities (§ 1) for the eigenvalues of the inertia matrix, which give here the fact that $2A \geq 1$, hence

$$A \in \left\{\frac{1}{2}, \frac{2}{3}, 1, 2\right\}.$$

The difference of the two roots of the first equation, namely $\sqrt{8A-7}$, can be an integer for A in this list only if

$$A \in \{1, 2\}.$$

In view of Haine's criterion, this concludes the proof of Proposition 3.1. □

3.4 More on the monodromy group

According to our program, let us compare this result with what the Morales–Ramis theorem tells us. To make things simpler, let us first notice that we do not need *two* poles. Obviously, the involution

$$\tau : (p, q, r, \gamma_1, \gamma_2, \gamma_3) \longmapsto (p, -q, -r, \gamma_1, -\gamma_2, -\gamma_3)$$

preserves W_0 and the total energy H. Its restriction to $\mathcal{E}_h$, still denoted τ,

$$(\gamma_1, q, \gamma_3) \longmapsto (\gamma_1, -q, -\gamma_3)$$

leaves the $\mathcal{E}_h$ and the system invariant. Moreover, it has no fixed point on the curve if $h \neq \pm 1$ (that is, when the curve is smooth). It exchanges the two points at infinity. Hence the quotient of our elliptic curve minus two points $\mathcal{E}_h$ by this involution is an elliptic curve minus one point $\mathcal{E}'_h$. The same is true of the restriction to $\mathcal{F}_h$, the quotient of which we will denote $\mathcal{F}'_h$ (this reduction was already used by Ziglin in [21]).

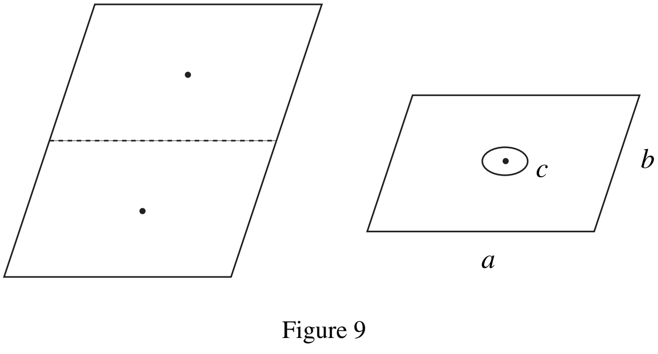


Figure 9

The fundamental group of $\mathcal{E}'_h$ is the free group on two generators a and b (Figure 9). When the monodromy about c is trivial (this is what we have discussed in Proposition 3.1), the monodromy group is Abelian. Notice that our variational equation has only regular singular points.

Corollary 3.3. *If the Galois groups of the variational equation along the curves $\mathcal{E}_h$ and $\mathcal{F}_h$ are both Abelian, then $A/C = 1$ or 2.* □

We have proved here that, in the case under consideration, the Kowalevski property implies that the Galois group is Abelian.

Unfortunately, this is not exactly what we want. Schematically, writing Gal for the Galois group and Gal° for its neutral component, what we have so far is:

$$\begin{array}{ccccc} (\mathrm{K}) & \Longrightarrow & (\mathrm{H}) & \Longleftrightarrow & \text{Gal Abelian} \\ & & & & \Downarrow \\ (\mathrm{L}) & \Longrightarrow & (\mathrm{MR}) & = & \text{Gal}^\circ \text{ Abelian} \end{array}$$

and we would like to understand whether the vertical arrow can be reverted.

3.5 More on the monodromy group – the case of the curve $\mathcal{E}_h$

To investigate Liouville integrability, we will need an additional argument:

Lemma 3.4. *There is a non empty open interval in* $]0, +\infty[$ *such that, for h in this interval, there is a cycle on the curve $\mathcal{E}'_h$ the monodromy along which has two real positive distinct eigenvalues.*

Accepting this for the moment, call a the monodromy along such a cycle. Let us call λ, λ^{-1} the two positive eigenvalues, chosen so that $0 < \lambda^{-1} < 1 < \lambda$. The algebraic subgroup of $\mathrm{SL}(2;\mathbb{C})$ generated by a is conjugated with

$$H = \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \mid z \in \mathbb{C}^\star \right\}.$$

An algebraic subgroup of $\mathrm{SL}(2;\mathbb{C})$ containing such a subgroup H and which is virtually Abelian is either Abelian or conjugated to a subgroup

$$G = \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \mid z \in \mathbb{C}^\star \right\} \cup \left\{ \begin{pmatrix} 0 & \mu^{-1} \\ -\mu & 0 \end{pmatrix} \mid \mu \in \mathbb{C}^\star \right\}$$

(the list of all algebraic subgroups of $\mathrm{SL}(2;\mathbb{C})$ is rather short and can be found, for instance, in [17]). Hence, either Gal is connected (and we are done) or it contains an element which writes, in a basis where a is diagonal, $\begin{pmatrix} 0 & \mu^{-1} \\ -\mu & 0 \end{pmatrix}$. The subgroup H is Abelian. The commutator of the two elements a and $\begin{pmatrix} 0 & \mu^{-1} \\ -\mu & 0 \end{pmatrix}$ is then $\begin{pmatrix} \lambda^2 & 0 \\ 0 & \lambda^{-2} \end{pmatrix}$. Our monodromy about the pole must be, at worst, such a commutator. It should thus have a real eigenvalue. But the indicial equation about the pole has real eigenvalues, so that the monodromy must have complex eigenvalues of modulus 1. A contradiction with the fact that $\lambda > 1$. Hence Gal is Abelian. The monodromy around the pole is trivial, so that $\sqrt{8A-7}$ must be an odd integer, namely

$$A = 1 + \frac{m(m+1)}{2} = 1, 2, 4, \dots.$$

Proof of Lemma 3.4 – calling to Liapounov for help. To apply this, we need to prove Lemma 3.4. To do so, I will use (at last!) the real structure of our elliptic curves $\mathcal{E}_h$ and a theorem of Liapounov (in [15], another Russian paper published in French, another paper published in Toulouse):

Theorem 3.5 (Liapounov [15]). *Let f be a periodic real function which is everywhere ≥ 0. Then the eigenvalues of the monodromy of the differential equation $\ddot{y} = f(t)y$ are real, positive and distinct.*

The proof of this theorem is rather simple and can be found in [11] (see also [6], a paper in which I have used it in a rather similar context, that of Lamé equations, elliptic curves again).

Proof of Lemma 3.4. Recall that the linear differential equation along $\mathcal{E}_h$ is

$$\ddot{R} + \frac{q\gamma_3}{\gamma_1}\dot{R} + \left(\frac{\gamma_3^2}{A\gamma_1} + \gamma_1\right)R = 0.$$

The first thing to do is to put this equation in the form $\ddot{y} = f(t)y$. But this is fairly classical and easy, just write

$$R = e^{\varphi}y, \text{ with } 2\dot{\varphi} + \frac{q\gamma_3}{\gamma_1} = 0,$$

to get the differential equation satisfied by y, namely, after a few lines of computation using the differential system satisfied by γ_1, γ_3 and q end the relations between these variables,

$$\ddot{y} = f(t)y \text{ with } f(t) = \frac{1}{2A\gamma_1^2}\left(2(1-A)\gamma_1^3 - h\gamma_1^2 + 3h\right).$$

Now it is easy to check that, on the real part of the curve $\mathcal{F}_h$ (namely for γ_1, γ_3 and q real), we have:

- if $A < 1$, f is positive when
$$0 \leq h \leq \inf\left\{9(1-A), \sqrt{\frac{3}{3-2A}}\right\};$$
- if $A > 1$, f is positive for
$$A - 1 \leq h \leq 9(A-1).$$

Hence we can apply Liapounov's theorem. □

Note that, using this argument, we have shown that, for this linear differential equation, Haine's and Morales–Ramis's criteria give the same result.

3.6 The case of the curve $\mathcal{F}_h$

This is slightly different in the case of the other curve. Recall however that, because of the Goryachev–Chaplygin case, we cannot expect Liouville integrability coincide with Kowalevski property. This is exactly what will appear in the investigation of the relation between Haine's and Morales–Ramis's criteria for the differential equation linearized along the curve $\mathcal{F}_h$.

In the Goryachev–Chaplygin case (namely, here, for $A = 4$), the indicial equation for our second differential equation (along $\mathcal{F}_h$) has roots $-1/2$ and $-3/2$, which gives eigenvalues -1 for the monodromy around the pole(s), a non trivial monodromy. Notice that the particular solution $\mathcal{F}_h$ we are considering lies in W_0, so that we are, indeed, in the Goryachev–Chaplygin case. There are non meromorphic solutions (as this can be deduced, for the original non linear system, from the fact that Kowalevskaya did not find this case, and as it was noticed by Ziglin [21, footnote p.13]). This is a case where the Galois group is not Abelian but should be virtually Abelian.

Our linear differential equation along the elliptic curve $\mathcal{F}_h$ is a Lamé equation. The monodromy and Galois groups of Lamé equations have been extensively studied (a summary of the results can be found in [17]). The results depend on the coefficients of the equation and on the elliptic curve itself. I was not able to find a direct argument (as the one provided by Liapounov's theorem above) to show that, with the information we have, Lemma 3.6, which ends the proof of Theorem 3.2, holds.

Lemma 3.6 (Maciejewski–Przybylska [16]). *Assume $A/B = B/C$ and $z_0 = 0$. If the Galois group of the variational equations along the curves $\mathcal{E}_h$ and $\mathcal{F}_h$ are virtually Abelian, then $A/C = 1$, 2 or 4.*

Remark. The non existence of a *real* meromorphic additional integral can be derived, along the lines suggested by Ziglin [22] (see also [6]), as shown in [16].

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Formal power series solutions of the heat equation in one spatial variable

Werner Balser

Abteilung Angewandte Analysis
Universität Ulm
89069 Ulm, Germany
email: `balser@mathematik.uni-ulm.de`

Abstract. In this article we investigate formal power series solutions of the heat equation in one spatial variable. In previous work of Lutz, Miyake, and Schäfke, resp. of W. Balser, solutions of a Cauchy problem have been shown to be k-summable in a direction d if, and only if, the initial condition satisfies a certain condition. Here, we investigate the initial value problem for the spatial variable, finding new results especially for the case when the initial values are Gevrey functions of order larger than one, so that the corresponding power series solution diverges.

Contents

1 Introduction

We shall be concerned with the heat equation in one spatial dimension, denoted as $u_t = u_{zz}$, allowing both the time variable t as well as the spatial variable z to be complex. Our main interest lies in power series solutions of this equation, and since they will, in general, not converge for any pair (t, z) with $tz \neq 0$, we shall speak of *formal solutions*, using the notation

$$\hat{u}(t, z) = \sum_{j,n=0}^{\infty} \frac{t^j z^n}{j!n!} u_{jn} = \sum_{j=0}^{\infty} \frac{t^j}{j!} \hat{u}_{j*}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \hat{u}_{*n}(t). \tag{1.1}$$

In particular we wish to prove what may be called *an asymptotic existence result*:

Given any power series solution, does there exist a solution of the heat equation that is holomorphic in some region in $\mathbb{C}^2$ and asymptotic to the formal one, as t and/or z tend to the origin?

A series (1.1) formally satisfies the heat equation if, and only if,

$$u_{j+1,n} = u_{j,n+2} \quad \text{for all } j, n \geq 0. \tag{1.2}$$

These relations say that among those indices j and n with $2j + n =: \nu$ fixed, one coefficient u_{jn} may be chosen arbitrarily, while all the others are then determined by (1.2). This observation leads to *two natural parametrizations* for the set of all formal solutions of the heat equation:

(a) Let one sequence $(\phi_n)_0^\infty$ of complex numbers, or, equivalently, one formal series $\hat{\phi}(z) = \sum_0^\infty \phi_n z^n/n!$, be given. Then there is a unique formal solution $\hat{u}(t, z)$ of the heat equation with coefficients u_{jn} satisfying $u_{0n} = \phi_n$ for all $n \geq 0$. This corresponds to solving the following *formal Cauchy problem*

$$u_t = u_{zz}, \quad \hat{u}(0, z) = \hat{\phi}(z). \tag{1.3}$$

The (unique) formal solution to this problem then is given by

$$\hat{u}(t, z) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \hat{\phi}^{(2j)}(z) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \sum_{n=0}^{\infty} \frac{z^n}{n!} \phi_{n+2j}. \tag{1.4}$$

Using the exponential-differential operator $\exp(t\partial_z^2) = \sum_{j=0}^\infty \partial_z^{(2j)} t^j/j!$, we can very elegantly write this formal series as

$$\hat{u}(t, z) = \exp(t\partial_z^2)\hat{\phi}(z).$$

Series of this form will in general diverge for every $t \neq 0$, even if the series $\hat{\phi}(z)$ converges, say for $|z| < r$, with $r > 0$. In an article of Lutz, Miyake, and Schäfke [8], 1-summability[1] in a direction d of such a series has been shown to be equivalent to some (explicit) condition upon the initial value function $\phi(z)$ defined by the convergent series $\hat{\phi}(z)$. In [3], an analogous result has been obtained for k-summability, with $k > 1$.

(b) Let two sequences (ψ_j), (η_j) of complex numbers, or, equivalently, two formal series $\hat{\psi}(t) = \sum_0^\infty \psi_j t^j/j!$, $\hat{\eta}(t) = \sum_0^\infty \eta_j t^j/j!$ be given. Then there is a unique formal solution $\hat{u}(t, z)$ of the heat equation with coefficients u_{jn} satisfying $u_{j0} = \psi_j$, $u_{j1} = \eta_j$, for all $j \geq 0$. This corresponds to solving the following *formal initial value problem* in the spatial variable z:

$$u_t = u_{zz}, \quad \hat{u}(t, 0) = \hat{\psi}(t), \quad \hat{u}_z(t, 0) = \hat{\eta}(t). \tag{1.5}$$

[1] For the exact definition of k-summability, or multisummability, or other terms used later, see, e.g., [4].

The (unique) formal solution to this problem then is given by

$$\left.\begin{aligned}\hat u(t,z) &= \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}\,\hat\psi^{(n)}(t) \;+\; \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}\,\hat\eta^{(n)}(t)\\ &= \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \sum_{j=0}^{\infty} \frac{t^j}{j!}\,\psi_{j+n}\\ &\qquad + \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \sum_{j=0}^{\infty} \frac{t^j}{j!}\,\eta_{j+n}.\end{aligned}\right\} \tag{1.6}$$

Similar to the situation in the first case, we can write this series as

$$\hat u(t,z) = \cosh(z\partial_t^{1/2})\hat\psi(t) + \sinh(z\partial_t^{1/2})\partial_t^{-1/2}\hat\eta(t),$$

with the usual interpretation of fractional differentiation resp. integration of formal power series in t. Concerning the question of convergence for series of this form, the situation here is essentially different from the one considered above: Whenever the series $\hat\psi(t)$ and $\hat\eta(t)$ both have a positive radius of convergence, then (1.6) converges for every $z \in \mathbb{C}$, and even for more general series, convergence of (1.6) takes place in some sense that shall be made clear later. Our main goal is to investigate the situation when even this is no longer true. This problem has been posed to the author by Th. Meurer [9]; also see [10], [11] for recent applications of k-summability to control problems.

In the following section we shall show that the two problems formulated above are formally related. Then we shall investigate the second problem in detail, before showing how these results relate to earlier ones on the first problem.

2 Formal transfer

In the introduction we have posed two natural problems concerning formal solutions of the heat equation. Interchanging the order of summation in (1.4) and comparing to (1.6), one finds that $\psi_j = \phi_{2j}$, $\eta_j = \phi_{2j+1}$ for every $j \ge 0$. For later use, it shall be of importance to express these relations in terms of the (formal) *acceleration resp. deceleration operators* introduced by J. Ecalle [5], [6], [7]; for their definition and the notation used here, refer to [4].

Lemma 1. (a) *Let $\hat\phi(z)$ be any formal power series, and let $\hat u(t,z)$ be the unique formal solution of* (1.3). *Then $\hat u(t,z)$ also solves* (1.5), *with $\hat\psi(t)$ and $\hat\eta(t)$ given by*

$$\hat\psi(t) = \frac{\hat h(t^{1/2}) + \hat h(-t^{1/2})}{2}, \quad \hat h(z) = \hat{\mathcal{A}}_{2,1}\big(\hat\phi(z)\big) = \sum_{j=0}^{\infty} \frac{t^j \phi_j}{\Gamma(1+j/2)},$$

$$\hat{\eta}(t) = \frac{\hat{\ell}(t^{1/2}) + \hat{\ell}(-t^{1/2})}{2}, \quad \hat{\ell}(z) = \hat{\mathcal{A}}_{2,1}\big(\hat{\phi}'(z)\big) = \sum_{j=0}^{\infty} \frac{t^j \phi_{j+1}}{\Gamma(1+j/2)}.$$

(b) *Let $\hat{\psi}(t)$ and $\hat{\eta}(t)$ be any formal power series, and let $\hat{u}(t,z)$ be the unique formal solution of* (1.5). *Then $\hat{u}(t,z)$ also solves* (1.3), *with $\hat{\phi}(z)$ given by*

$$\hat{\phi}(z) = \hat{h}_+(z^2) + \int_0^z \hat{h}_-(\zeta^2)\, d\zeta,$$

$$\hat{h}_+(t) = \hat{\mathcal{D}}_{1,1/2}(\hat{\psi}(t)), \quad \hat{h}_-(t) = \hat{\mathcal{D}}_{1,1/2}(\hat{\eta}(t)).$$

Proof. Recall from [4] that

$$\hat{\mathcal{A}}_{2,1} \frac{t^\alpha}{\Gamma(1+\alpha)} = \frac{t^\alpha}{\Gamma(1+\alpha/2)}, \quad \hat{\mathcal{D}}_{1,1/2} \frac{t^\alpha}{\Gamma(1+\alpha)} = \frac{t^\alpha}{\Gamma(1+2\alpha)} \quad \text{for all } \alpha \ge 0.$$

Using this and the fact that formal acceleration and deceleration operators, by definition, are applied term by term to formal power series, one can complete the proof. □

The lemma stated above shows that the two initial value problems (1.3) and (1.5) are *formally equivalent*, and what we shall do in the final section is to show that, in some sense, they are holomorphically equivalent, too. This, however, needs to be formulated in a more precise manner.

3 The initial value problem

In this section we shall investigate the initial value problem (1.5), but with initial conditions $\psi(t)$ and $\eta(t)$ that are functions which are holomorphic in a sectorial region G and have the formal power series $\hat{\psi}(t)$ and $\hat{\eta}(t)$ as their asymptotic expansion of some Gevrey order $s \ge 0$. In our investigation, we shall make use of the entire function given by the power series

$$k(w) = \sum_{n=0}^{\infty} \frac{n!}{(2n)!} w^n \quad \text{for all } w \in \mathbb{C}. \tag{3.1}$$

We shall refer to $k(w)$ as the *kernel function* for the complex heat equation, and we show that it has the following properties:

Lemma 2. *The kernel function $k(w)$ is entire and of exponential order* 1 *and finite type. More precisely, there exists a constant $c > 0$ such that*

$$|k(w)| \le c|w|^{1/2} \exp[\Re w/4] \quad \textit{for all } w \in \mathbb{C} \textit{ with } \Re w \ge 0. \tag{3.2}$$

In the left halfplane, the function $k(w)$ remains bounded; more precisely, in the open sector $S_- = \{w \in \mathbb{C} \setminus \{0\} : \pi/2 < \arg w < 3\pi/2\}$ one has

$$wk(w) \to 2 \quad \text{as } w \to \infty \text{ in } S_-. \tag{3.3}$$

In addition, $k(w)$ is a solution of the following first order ordinary differential equation:

$$4wk' - (2+w)k = -2. \tag{3.4}$$

Proof. The differential equation (3.4) can be verified directly, using the power series representation for $k(w)$. The remaining statements of the lemma can then be proven solving (3.4) by variation of constants and integration by parts. An alternative proof is using the following representation formula for $k(w)$: Since $(2n)!/n! = 4^n (1/2)_n = 4^n \Gamma(n+1/2)/\Gamma(1/2)$, one can use the standard integral representation of the reciprocal Gamma function (see, e.g., [4, p. 228]) to obtain

$$k(w) = \frac{\sqrt{\pi}}{2\pi i} \int_{\gamma_R} \mathrm{e}^u \frac{u^{1/2}}{u - w/4} du, \quad |w| < R, \tag{3.5}$$

with $R > 0$ chosen arbitrarily and a path γ_R from ∞ along the ray $\arg u = -\pi$ to the point $u = -R$, then along the positively oriented circle $|u| = R$ to the same point, and back to ∞ along $\arg u = \pi$ (and choosing the branch of $u^{1/2}$ accordingly). Using residue calculus, very much like in the proof of [4, Lemma 6, p. 84], one can show that $k(w) = \sqrt{\pi}\mathrm{e}^{w/4}(w/4)^{1/2} + \tilde{k}(w)$, with $\tilde{k}(w)$ given by the same integral representation (3.5), but for $|w| > R$. This representation of $\tilde{k}(w)$ shows that $w\tilde{k}(w) \to \sqrt{\pi}(-4)/\Gamma(-1/2) = 2$, which implies (3.2) and (3.3). □

With help of this kernel function, we now show:

Theorem 1. *Let $G \subset \mathbb{C}$ be any region, and let $\psi(t)$, $\eta(t)$ be holomorphic in G. Then for $t \in G$, $z \in \mathbb{C}$, and $\varepsilon > 0$ so that the circle of radius ε about the point t belongs to G, the function*

$$\begin{aligned} u(t,z) = {} & \frac{1}{2\pi i} \oint_{|\tau - t| = \varepsilon} k\big(z^2/(\tau - t)\big) \frac{\psi(\tau)}{\tau - t} d\tau \\ & + \int_0^z \frac{1}{2\pi i} \oint_{|\tau - t| = \varepsilon} k\big(\zeta^2/(\tau - t)\big) \frac{\eta(\tau)}{\tau - t} d\tau d\zeta \end{aligned} \tag{3.6}$$

is the unique solution of the initial value problem

$$u_t = u_{zz}, \quad u(t,0) = \psi(t), \quad u_z(t,0) = \eta(t).$$

This solution is an entire function in the variable z, whose power series expansion, for fixed $t \in G$, is given by

$$u(t,z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \psi^{(n)}(t) + \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \eta^{(n)}(t) \quad \text{for all } z \in \mathbb{C}. \tag{3.7}$$

Proof. One can use either (3.4) or the power series for $k(w)$ to show that the function $(\tau - t)^{-1}k(z^2(\tau - t)^{-1})$, for fixed $\tau \in \mathbb{C}$ and $t, z \in \mathbb{C}$ with $t \neq \tau$, is a solution of the heat equation. Observing that partial differentiation under the integral signs in (3.6) is permitted, use this and termwise integration of the expansion of $k(z^2/(\tau - t))$ to complete the proof. □

The above theorem may be used as follows: Let two formal series $\hat{\psi}(t)$ and $\hat{\eta}(t)$ be given. We select an arbitrary sectorial region G and two functions $\psi(t)$ and $\eta(t)$ that are holomorphic in G and asymptotic to $\hat{\psi}(t)$ resp. $\hat{\eta}(t)$, as $t \to 0$ in G; existence of such functions follows from *Ritt's Theorem* (see, e.g., [4, Theorem 16, p. 68]). Corresponding to these two functions, we can define the solution (3.6), and in this way, to every pair of formal series $\hat{\psi}(t)$ and $\hat{\eta}(t)$ there corresponds a whole class of solutions $u(t, z)$. It remains to investigate in which way these solutions are asymptotically related to the two formal series with which we started, and in which cases we may choose *one* of these solutions, say, having the most intimate relation to $\hat{\psi}(t)$, $\hat{\eta}(t)$, and award it the title of *sum* of the formal solution (1.6). This investigation is different in the following cases:

Proposition 1. *Given formal power series $\hat{\psi}(t)$ and $\hat{\eta}(t)$ and a sectorial region G, assume existence of functions $\psi(t)$ and $\eta(t)$ that are holomorphic in G and asymptotic to $\hat{\psi}(t)$ resp. $\hat{\eta}(t)$ of Gevrey order s, with* $0 \le s < 1$. *Then the series*

$$\phi(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}\psi_n + \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}\eta_n \tag{3.8}$$

converges for all $z \in \mathbb{C}$, and $\phi(z)$ is an entire function of exponential order $\kappa = 2/(1 - s)$ and finite type. Moreover, rewriting the formal solution (1.6) *as $\hat{u}(t, z) = \sum_{j=0}^{\infty} t^j \phi^{(j)}(z)/j!$, the solution* (3.6), *for every $z \in \mathbb{C}$, is asymptotic to $\hat{x}(t, z)$ of Gevrey order s, as $t \to 0$ in G, and the asymptotic expansion is locally uniform in z.*

Proof. Let $\overline{S}$ be any closed subsector of G. Then the general theory of Gevrey expansions implies existence of constants $C, K > 0$ so that

$$|\psi^{(n)}(t)|, |\eta^{(n)}(t)| \le C K^n n! \Gamma(1 + sn) \quad \text{for all } t \in \overline{S}. \tag{3.9}$$

Accordingly, convergence of (3.8) follows, owing to $\psi^{(n)}(0) = \psi_n$, $\eta^{(n)}(0) = \eta_n$. Moreover, we obtain from (3.7) that for $(z, t) \in \mathbb{C} \times \overline{S}$ and $m \in \mathbb{N}_0$

$$|\partial_t^m u(t, z)| \le C\Big[\sum_{n=0}^{\infty} \Big\{\frac{|z|^{2n}}{(2n)!} + \frac{|z|^{2n+1}}{(2n+1)!}\Big\} K^{(n+m)}(n+m)!\Gamma(1 + s(n+m))\Big]$$

which, with help of Stirling's formula, implies for every $R > 0$ existence of constants $C, K > 0$ that may be different from above, so that

$$|\partial_t^m u(t, z)| \le C K^m m! \Gamma(1 + sm) \quad \text{for all } |z| \le R, t \in \overline{S}, m \in \mathbb{N}_0.$$

Finally, one can check that $\partial_t^m u(t,z) \to \phi^{(2m)}(z)$ as $t \to 0$ in G. This completes the proof. □

The proof of the last result is clearly based upon the fact that the series (3.7) converges uniformly in t, for t in closed subsectors of G and arbitrary $z \in \mathbb{C}$. This convergence fails for $s = 1$, if $|z|$ is too large. So for this case we show:

Proposition 2. *Given formal power series $\hat{\psi}(t)$ and $\hat{\eta}(t)$ and a sectorial region G, assume existence of functions $\psi(t)$ and $\eta(t)$ that are holomorphic in G and asymptotic to $\hat{\psi}(t)$ resp. $\hat{\eta}(t)$ of Gevrey order s, with $s = 1$. Then there exists a $\rho > 0$ so that the series* (3.8) *converges for $|z| < \rho$. Moreover, with $\hat{u}(t,z)$ as in Proposition* 1*, for every closed subsector $\overline{S} \subset G$ there exists an $R > 0$ so that for $|z| < R$ the solution* (3.6) *is asymptotic to $\hat{x}(t,z)$ of Gevrey order s, as $t \to 0$ in $\overline{S}$, and the asymptotic expansion is uniform in z. Finally, if G has bisecting direction d and opening $\alpha > \pi$, then $u(t,z)$ also is asymptotic to $\hat{u}(t,z)$ as $t \to 0$, for z, t such that*

$$|d - 2\arg z| < (\alpha - \pi)/2, \quad |d - \arg t| < \pi/2, \quad |t| < r, \tag{3.10}$$

with sufficiently small $r > 0$, but no restriction upon $|z|$.

Proof. The first assertions follow as in the proof of the previous proposition. For the remaining statement, observe that for $m \in \mathbb{N}$ the function $\tilde{u}(t,z) = \partial_t^m u(t,z)$ is the unique solution of the heat equation with respect to the initial conditions $\tilde{u}(t,0) = \psi^{(m)}(t)$, $\tilde{u}_z(t,0) = \eta^{(m)}(t)$ and hence has a representation analogous to (3.6), but with $\psi^{(m)}(t)$, $\eta^{(m)}(t)$ in place of $\psi(t)$, $\eta(t)$. In this formula we may make a change of variable $\tau - t = u$ and, instead of a circle, integrate along a curve γ as follows: From the origin along the ray $\arg u = 2\arg z - \delta - \pi/2$, with small $\delta > 0$, to a point of sufficiently small modulus, then along a circular arc to the ray $\arg u = 2\arg z + \delta + \pi/2$, and back to the origin along this ray. For z and t as in (3.10), and δ small enough, this path is such that z^2/u remains in the left halfplane as $t \to 0$, and so $k(z^2/(\tau - t))$ is bounded, owing to Lemma 2, with a bound that is independent of t and z. At the same time, $u + t$ remains within the region G. Using this, we conclude that for all $m \in \mathbb{N}_0$ and t, z as in (3.10)

$$\begin{aligned}
\partial^m u(t,z) &= \frac{1}{2\pi i}\int_\gamma k\big(z^2/u\big)\frac{\psi^{(m)}(u+t)}{u}du \\
&\quad + \int_0^z \frac{1}{2\pi i}\int_\gamma k\big(\zeta^2/u\big)\frac{\eta^{(m)}(u+t)}{u}du\,d\zeta \\
&\to \frac{1}{2\pi i}\int_\gamma k\big(z^2/\tau\big)\frac{\psi^{(m)}(\tau)}{\tau}d\tau \\
&\quad + \int_0^z \frac{1}{2\pi i}\int_\gamma k\big(\zeta^2/\tau\big)\frac{\eta^{(m)}(\tau)}{\tau}d\tau\,d\zeta =: \phi_m(z),
\end{aligned} \tag{3.11}$$

as $t \to 0$. From the general theory on integral operators in [4, Chapter 5], one finds that $\phi_0(z) = \phi(z)$, and $\phi_m(z) = \phi^{(2m)}(z)$, $m \ge 0$. This completes the proof. □

As the most interesting case, we now treat the situation of initial conditions $\psi(t)$, $\eta(t)$ that have Gevrey expansions of order $s > 1$, or even asymptotic expansions in the classical sense:

Proposition 3. *Given formal power series $\hat{\psi}(t)$ and $\hat{\eta}(t)$ and a sectorial region G of bisecting direction d and opening $\alpha > \pi$, let functions $\psi(t)$ and $\eta(t)$ be holomorphic in G and asymptotic to $\hat{\psi}(t)$ resp. $\hat{\eta}(t)$ as $t \to 0$. Rewriting the formal solution* (1.6) *as $\hat{u}(t, z) = \sum_{j=0}^{\infty} t^j \phi^{(j)}(z)/j!$, with $\phi(z) = \phi_0(z)$ defined as in* (3.11)*, then $u(t, z)$ is asymptotic to $\hat{u}(t, z)$ as $t \to 0$, for z, t as in* (3.10)*, with no restriction upon $|z|$. If the expansions of $\hat{\psi}(t)$ and $\hat{\eta}(t)$ both are of Gevrey order $s > 1$, then the same holds true for the expansion of $u(t, z)$, locally uniformly in the variable z.*

The proof of this result is exactly the same as the second part of the proof of the previous proposition and is, therefore, not repeated here.

4 Concluding remarks

The results of the previous section may be applied as follows to reprove results from [8], [3] and obtain some new ones:

(1) Let formal power series $\hat{\psi}(t)$ and $\hat{\eta}(t)$ be given. According to Ritt's theorem [4, p. 68], for every sector S of finite opening, there exist functions $\psi(t)$ and $\eta(t)$ that are asymptotic to the formal series as $t \to 0$ in S. These functions, however, are not unique! Nonetheless, to every pair of such functions, there corresponds a solution of the heat equation defined by (3.6), which is entire in z and holomorphic for $t \in S$. If the opening of S is larger than π (and we may always assume this to hold, since in Ritt's theorem no restriction upon the opening is made), then for t and z as in (3.10) this solution, owing to Proposition 3, is asymptotic to the formal solution (1.6), rewritten as $\hat{u}(t, z) = \sum_j \phi^{(2j)} t^j / j!$, as $t \to 0$, with $\phi(z) = \phi_0(z)$ defined by (3.11). It is worth observing that (3.11) may be regarded as the analogue to the formal transfer from (1.5) to (1.3) that was made in Lemma 1 (b), and that $\phi(z)$ is asymptotic to the formal series $\hat{\phi}(z)$ defined there. However, if $\hat{\phi}(z)$ happens to converge, the function $\phi(z)$ will, in general, still not be holomorphic at the origin, hence will in such cases not coincide with the sum of $\hat{\phi}(z)$.

(2) Let $k_1 > \cdots > k_p > 0$ be given, and let $\hat{\psi}(t)$, $\hat{\eta}(t)$ be $(k_1, \ldots, k_p)$-summable in an admissible multidirection $(d_1, \ldots, d_p)$. Then, their sums $\psi(t)$, $\eta(t)$ are holomorphic in a sectorial region G of bisecting direction d_1 and opening larger than π/k_1. Moreover, $\psi(t)$, $\eta(t)$ are asymptotic to $\hat{\psi}(t)$, $\hat{\eta}(t)$ of Gevrey order $s = 1/k_p$, as $t \to 0$ in G. The solution $u(t, z)$ corresponding to the sums $\psi(t)$, $\eta(t)$ then is,

in fact, *uniquely determined by the two formal series* $\hat{\psi}(t)$, $\hat{\eta}(t)$ (or equivalently, by the formal solution(1.6)), and we shall therefore consider it as *the sum of the formal solution* (1.6). Comparing with existing results, the following cases occur:

(a) If $k_p \geq 1$, the function $\phi(z)$ defined by (3.8) is holomorphic near the origin, and even is entire for $k_p > 1$. Rewriting (1.6) in the form $\hat{u}(t,z) = \sum_j \phi^{(2j)}(z)t^j/j!$, one can show that this series also is $(k_1,\ldots,k_p)$-summable in the multidirection $(d_1,\ldots,d_p)$, and conversely, $(k_1,\ldots,k_p)$-summability of $\hat{u}(t,z) = \sum_j \phi^{(2j)}(z)t^j/j!$ in the multidirection $(d_1,\ldots,d_p)$ implies the same for $\hat{\psi}(t)$, $\hat{\eta}(t)$. For $p = 1$ and $k_1 = 1$, this coincides with the result of Lutz, Miyake, and Schäfke [8], while the general case has been treated in [3].

(b) For $k_1 \leq 1$, the sum $u(t,z)$ still is asymptotic to the formal solution $\hat{u}(t,z)$, but in a sector that is too small for any type of multisummability of $\hat{u}(t,z)$. These cases have not been studied before.

(c) According to [1], [2], multisummable series can be decomposed into a finite sum of series that are k-summable for a scalar $k > 0$. Accordingly, one can deduce from above the corresponding asymptotic properties of the sum $u(t,z)$ corresponding to situations with $k_1 > 1 > k_p$.

Altogether, we have shown in this article that to every formal power series solution of the heat equation there exist holomorphic solutions which are asymptotically related to the formal one. Parametrizing the formal solution as in (1.6), one even has a unique holomorphic solution whenever the series $\hat{\psi}(t)$ and $\hat{\eta}(t)$ are multisummable.

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Multiplicity of critical points of master functions and Schubert calculus

*Prakash Belkale, Evgeny Mukhin and Alexander Varchenko**

Department of Mathematics
University of North Carolina at Chapel Hill, U.S.A.
email: `belkale@email.unc.edu`

Department of Mathematical Sciences
Indiana University, Purdue University, Indianapolis, U.S.A.
email: `mukhin@math.iupui.edu`

Department of Mathematics
University of North Carolina at Chapel Hill, U.S.A.
email: `anv@email.unc.edu`

Abstract. In [MV], some correspondences were defined between critical points of master functions associated to $\mathfrak{sl}_{N+1}$ and subspaces of $\mathbb{C}[x]$ with given ramification properties. In this paper we show that these correspondences are in fact scheme theoretic isomorphisms of appropriate schemes. This gives relations between multiplicities of critical point loci of the relevant master functions and multiplicities in Schubert calculus.

Contents

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1 Introduction

In [MV], a correspondence between the following objects was shown:

(1) critical points of certain master functions associated to the Lie algebra $\mathfrak{sl}_{N+1}$;

(2) vector subspaces $V \subset \mathbb{C}[x]$ of rank $N+1$ with prescribed ramification at points of $\mathbb{C} \cup \{\infty\}$.

Let $l_1, \dots, l_N$ be nonnegative integers and $z_1, \dots, z_n$ distinct complex numbers. For $s = 1, \dots, n$, fix nonnegative integers $m_s(1), \dots, m_s(N)$. Define polynomials $T_1, \dots, T_N$ by the formula $T_i = \prod_{s=1}^{n} (x - z_s)^{m_s(i)}$. *The master function* Φ *associated to this data* is the rational function

$$\Phi(t_j^{(i)}) = \prod_{i=1}^{N}\prod_{j=1}^{l_i} T_i(t_j^{(i)})^{-1} \prod_{i=1}^{N-1}\prod_{j=1}^{l_i}\prod_{k=1}^{l_{i+1}} (t_j^{(i)} - t_k^{(i+1)})^{-1} \prod_{i=1}^{N}\prod_{1\le j<k\le l_i} (t_j^{(i)} - t_k^{(i)})^2$$

of $l_1 + \cdots + l_N$ variables $t_1^{(1)}, \dots, t_{l_1}^{(1)}, \dots, t_1^{(N)}, \dots, t_{l_N}^{(N)}$ considered on the set of points where

- the numbers $t_1^{(i)}, \dots, t_{l_i}^{(i)}$ are distinct;
- the sets $\{t_1^{(i)}, \dots, t_{l_i}^{(i)}\}$ and $\{t_1^{(i+1)}, \dots, t_{l_{i+1}}^{(i+1)}\}$ do not intersect;
- the sets $\{t_1^{(i)}, \dots, t_{l_i}^{(i)}\}$ and $\{z_1, \dots, z_n\}$ do not intersect.

The master functions considered in (1) are functions Φ as above. These functions appear in hypergeometric solutions to the KZ equation. They also appear in the Bethe ansatz method of the Gaudin model, where the goal is to write formulas for singular vectors in a tensor product of representations of $\mathfrak{sl}_{N+1}$ starting from a critical point. Those Bethe vectors are eigenvectors of certain commuting linear operators called Hamiltonians of the Gaudin model. We refer the reader to [V] for a detailed discussion of these themes.

The set of objects given by (2) is important in the study of linear series on compact Riemann surfaces (see [EH]).

In both of the objects (1), (2) above, there is a natural notion of multiplicity. In (1), we can consider the geometric multiplicity of the critical scheme (see Proposition 6.7). In (2), we may view the set of such objects as the intersection of some Schubert cells in a Grassmannian $\mathrm{Gr}(N+1, \mathbb{C}_d[x])$ and hence, there is an associated intersection multiplicity (which is in this case the same as the geometric multiplicity at the given V

of the intersection of Schubert cells). In this paper, we show that these multiplicities agree (Theorems 2.7, 4.2, and corollaries 2.8, 4.4).

One consequence of this agreement is that intersection numbers in Grassmannians can be calculated from the critical scheme of master functions and vice versa. As indicated in [MV], this relation provides a link via critical schemes of master functions, between representation theory of $\mathfrak{sl}_{N+1}$ and Schubert calculus (see [B] for a different geometric approach).

A related consequence is that the intersection of associated Schubert cells is transverse at V if and only if the associated critical scheme is of geometric multiplicity 1.

A note on our methods: we show the equalities of multiplicities by using Grothendieck's scheme theory. To obtain statement on multiplicities, we need to show that the correspondences are in fact isomorphisms of (appropriate) schemes. By Grothendieck's functorial approach to schemes, this aim will be achieved if we can replace $\mathbb{C}$ in [MV] by an arbitrary local ring A and show that the correspondences hold with objects over A.

This requires us to develop the theory of Wronskian equations over an arbitrary local ring, in particular to develop criteria for solvability in a purely algebraic manner (see Lemma 6.1). We also need to revisit key arguments in [MV] and modify their proof so that they apply over any local ring (see Theorem 3.5).

1.1 Notation

For a ring A, let $A_d[x]$ denote the set of polynomials with coefficients in A of degree $\leq d$. An element in $A[x]$ is said to be monic, if its leading coefficient is invertible. The multiplicative group of units in A is denoted by A^*.

For a ring A and elements $y_1, \dots, y_k \in A$, we will denote by $(y_1, \dots, y_k)$ the ideal generated by the y_i in A.

A local ring $(A, \mathfrak{m})$ over $\mathbb{C}$ is a Noetherian ring A containing $\mathbb{C}$ with a unique maximal ideal $\mathfrak{m}$. The residue field of the local ring is defined to be $A/\mathfrak{m}$. We will only consider local rings containing $\mathbb{C}$ with residue field $\mathbb{C}$. In this paper, a scheme stands for an algebraic scheme over $\mathbb{C}$.

We denote the permutation group of the set $\{1, \dots, N\}$ by Σ_N.

2 Formulation of the main result

2.1 Some preliminaries

For a vector bundle $\mathcal{W}$ on a scheme X, denote by $\mathrm{Fl}(\mathcal{W}) \to X$ the fiber bundle whose fiber over a point $x \in X$ is the flag variety of complete filtrations of the fiber $\mathcal{W}_x$.

Let W be a vector space of dimension $d+1$ and N, $0 \le N \le d$, an integer. There is a natural exact sequence of vector bundles on the Grassmannian $\mathrm{Gr}(N+1, W)$,

$$0 \longrightarrow \mathcal{V} \longrightarrow W \otimes \mathcal{O} \longrightarrow \mathcal{Q} \longrightarrow 0.$$

The fiber of this sequence at a point $V \in \mathrm{Gr}(N+1, W)$ is

$$0 \longrightarrow V \longrightarrow W \longrightarrow W/V \longrightarrow 0.$$

Let $\mathcal{F}$ be a complete flag on W:

$$\mathcal{F} : \{0\} = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_{d+1} = W.$$

A *ramification sequence* is a sequence $\boldsymbol{a}$ of the form $(a_1, \ldots, a_k) \in \mathbb{Z}^k$ such that $a_1 \ge \cdots \ge a_k \ge 0$. For a ramification sequence $\boldsymbol{a} = (a_1, a_2, \ldots, a_{N+1})$ satisfying $a_1 \le d - N$, define the Schubert cell

$$\Omega^o_{\boldsymbol{a}}(\mathcal{F}) = \{V \in \mathrm{Gr}(N+1, W) \mid \mathrm{rk}(V \cap F_u) = \ell,$$
$$d - N + \ell - a_\ell \le u < d - N + \ell + 1 - a_{\ell+1},\ \ell = 0, \ldots, N+1\},$$

where $a_0 = d - N$, $a_{N+2} = 0$. The cell $\Omega^o_{\boldsymbol{a}}(\mathcal{F})$ is a smooth connected variety. The closure of $\Omega^o_{\boldsymbol{a}}(\mathcal{F})$ is denoted by $\Omega_{\boldsymbol{a}}(\mathcal{F})$. The codimension of $\Omega^o_{\boldsymbol{a}}(\mathcal{F})$ is

$$|\boldsymbol{a}| = a_1 + a_2 + \cdots + a_{N+1}.$$

Every $N+1$-dimensional vector subspace of W belongs to a unique Schubert cell $\Omega^o_{\boldsymbol{a}}(\mathcal{F})$.

Denote by $\mathcal{V}_{\boldsymbol{a}}$ the pull-back of $\mathcal{V}$ to $\Omega^o_{\boldsymbol{a}}(\mathcal{F}) \hookrightarrow \mathrm{Gr}(N+1, W)$. There is *the distinguished section* of

$$\mathrm{Fl}(\mathcal{V}_{\boldsymbol{a}}) \longrightarrow \Omega^o_{\boldsymbol{a}}(\mathcal{F})$$

which assigns to a point $V \in \Omega^o_{\boldsymbol{a}}(\mathcal{F})$ the complete filtration

$$0 \subsetneq F_{d-N+1-a_1} \cap V \subsetneq F_{d-N+2-a_2} \cap V \subsetneq \cdots \subsetneq F_{d-N+N+1-a_{N+1}} \cap V = V.$$

This section may be used to partition each fiber $\mathrm{Fl}((\mathcal{V}_{\boldsymbol{a}})_V)$ into Schubert cells (see definitions in Section 2.2). This partition varies algebraically with V. That is, there is a decomposition of $\mathrm{Fl}(\mathcal{V}_{\boldsymbol{a}})$ into relative Schubert cells, each of which is a locally trivial (in the Zariski topology) fiber bundle over $\Omega^o_{\boldsymbol{a}}(\mathcal{F})$.

2.2 Schubert cells in flag varieties

Let V be a vector space of rank $N+1$, $\mathcal{F}$ a complete flag on V and $w \in \Sigma_{N+1}$. Define the Schubert cell $G^o_w(\mathcal{F})$ corresponding to w to be the subset of $\mathrm{Fl}(V)$ formed by points $V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_{N+1} = V$ such that there exists a basis $u_1, \ldots, u_{N+1}$ of V satisfying the conditions

$$V_i = \mathrm{Span}_{\mathbb{C}}(u_1, \ldots u_i), \quad u_i \in F_{w(i)},\ i = 1, \ldots, N+1.$$

It is easy to see that

$$\mathrm{Fl}(V) = \bigsqcup_{w \in \Sigma_{N+1}} G_w^o(\mathcal{F}).$$

It is easy to see that the permutation of highest length gives the *open cell* in this partition of $\mathrm{Fl}(V)$.

2.3 Intersection theory in spaces of functions

Let $W = \mathbb{C}_d[x]$ be the space of polynomials of degree not greater than d. Each point $z \in \mathbb{C} \cup \{\infty\}$ determines a full flag in W:

$$\mathcal{F}(z) : 0 = F_0(z) \subsetneq F_1(z) \subsetneq \cdots \subsetneq F_{d+1}(z) = W$$

where for any $z \in \mathbb{C}$ and any i, $F_i(z) = (x - z)^{d+1-i}\mathbb{C}[x] \cap W$, and if $z = \infty$, then $F_i(z)$ is the space of polynomials of degree $< i$.

For $V \in \mathrm{Gr}(N + 1, W)$ and $z \in \mathbb{C} \cup \{\infty\}$, there exists a unique ramification sequence $\boldsymbol{a}(z) = (a_1, \dots, a_{N+1})$, with $a_1 \le d - N$, such that $V \in \Omega^o_{\boldsymbol{a}(z)}(\mathcal{F}(z))$. The sequence is called *the ramification sequence of V at z.*

If $z \in \mathbb{C}$, then this means that V has a basis of the form

$$(x - z)^{N+1-1+a_1} f_1,\ (x - z)^{N+1-2+a_2} f_2, \dots,\ (x - z)^{N+1-(N+1)+a_{N+1}} f_{N+1}$$

with $f_i(z) \ne 0$ for $i = 1, \dots, N + 1$. The numbers

$$\{N + 1 - i + a_i \mid i = 1, \dots, N + 1\}$$

are called the *exponents* of V at z.

If $z = \infty$, then the condition is that V has a basis of the form $f_1, f_2, \dots, f_{N+1}$ with $\deg f_i = d-(N+1)+i-a_i$ for $i = 1, \dots, N+1$. The numbers $\{d-(N+1)+i-a_i \mid i = 1, \dots, N + 1\}$, are called the *exponents* of V at ∞.

Remark 2.1. The set of exponents of V at any point in $\mathbb{C}\cup\{\infty\}$ is a subset of $\{0, \dots, d\}$ of cardinality $N + 1$.

A point $z \in \mathbb{C} \cup \{\infty\}$ is called *a ramification point of V* if $\boldsymbol{a}(z)$ is a sequence with at least one nonzero term.

For a finite dimensional subspace $E \subseteq \mathbb{C}[x]$, define the Wronskian $\mathrm{Wr}(E) \in \mathbb{C}[x]/\mathbb{C}^*$ as the Wronskian of a basis of E. The Wronskian of a subspace is a nonzero polynomial with the following properties.

Lemma 2.2. *If $V \subseteq W$ lies in the Schubert cell $\Omega^o_{\boldsymbol{a}}(\mathcal{F}(z)) \subseteq \mathrm{Gr}(N+1, W)$ for some $z \in \mathbb{C}$, then $\mathrm{Wr}(V)$ has a root at z of multiplicity $|\boldsymbol{a}|$.*

If $V \in \Omega^o_{\boldsymbol{a}}(\mathcal{F}(\infty))$, then $\deg \mathrm{Wr}(V) = (N + 1)(d - N) - |\boldsymbol{a}|$.

We will fix the following objects:

- a space of polynomials $W = \mathbb{C}_d[x]$;
- a Grassmannian $\mathrm{Gr}(N+1, W)$ with the universal subbundle $\mathcal{V}$;
- distinct points $z_1, \dots, z_n$ on $\mathbb{C}$;
- at each point z_s, a ramification sequence $\boldsymbol{a}(z_s)$ and at ∞, a ramification sequence $\boldsymbol{a}(\infty)$ so that
$$\sum_{z \in \{z_1,\dots,z_n\}} |\boldsymbol{a}(z)| + |\boldsymbol{a}(\infty)| = \dim \mathrm{Gr}(N+1, W) = (N+1)(d-N). \quad (2.1)$$

We set

$$\Omega = \bigcap_{s=1}^{n} \Omega^{o}_{\boldsymbol{a}(z_s)}(\mathcal{F}(z_s)) \cap \Omega^{o}_{a(\infty)}(\mathcal{F}(\infty)),$$

$$K_i = \prod_{s=1}^{n} (x - z_s)^{\sum_{\ell=1}^{i} a_{N+1-i+\ell}(z_s)}, \quad i = 1, \dots, N+1,$$

$$T_i = K_{i+1}K_{i-1}/K_i^2 = \prod_{s=1}^{n} (x - z_s)^{a_{N+1-i}(z_s) - a_{N+2-i}(z_s)}, \quad i = 1, \dots, N.$$

We set $K_0 = 1$. Notice that T_i is a polynomial.

The collection of these objects will be called the "*basic situation*".

Remark 2.3. The following are basic facts from intersection theory in the space of polynomials.

- Ω is a finite scheme, its support is a finite set.
- In the definition of Ω, we intersected Schubert cells. We obtain the same intersection if we intersect the closures of the same Schubert cells:
$$\Omega = \bigcap_{s=1}^{n} \Omega_{\boldsymbol{a}(z_s)}(\mathcal{F}(z_s)) \cap \Omega_{\boldsymbol{a}(\infty)}(\mathcal{F}(\infty)).$$

Define Fl as a pull-back

$$\begin{array}{ccc} \mathrm{Fl} & \longrightarrow & \mathrm{Fl}(\mathcal{V}) \\ \pi \downarrow & & \downarrow \\ \Omega & \longrightarrow & \mathrm{Gr}(N+1, W) \end{array} \quad (2.2)$$

Points of Fl are pairs $(V, E_\bullet)$ where $V \in \Omega$ and $E_\bullet = (E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_{N+1} = V)$ is a complete filtration of V. For $V \in \Omega$, there are $n+1$ *distinguished complete filtrations on* V corresponding respectively to the flags $\mathcal{F}(z_1), \mathcal{F}(z_2), \dots, \mathcal{F}(z_n)$, and $\mathcal{F}(\infty)$ of W.

Let U be the open subset of Fl formed by points $(V, E_\bullet)$ such that $E_\bullet$ lies in the intersection of $n+1$ open Schubert cells corresponding respectively to the $n+1$

distinguished complete filtrations on V. This condition on $E_\bullet$ is equivalent to the statement that for $i = 1, \dots, N+1$ and $z \in \{z_1, \dots, z_n\}$, the subspace E_i has ramification sequence $(a_{N+1-i+1}(z), \dots, a_{N+1}(z))$ at z.

The subset U is dense in each fiber of $\mathrm{Fl} \to \Omega$.

Consider the subset $\mathrm{Fl}^o \subseteq U$ formed by points $(V, E_\bullet)$ such that for $i = 1, \dots, N-1$, the subspaces $E_i \subset \mathbb{C}[x]$ and $E_{i+1} \subset \mathbb{C}[x]$ do not have common ramification points in $\mathbb{C} - \{z_1, \dots, z_n\}$, i.e., their Wronskians do not have common roots in $\mathbb{C} - \{z_1, \dots, z_n\}$.

Lemma 2.4. Fl^o *is open and dense in each fiber of* $U \to \Omega$.

Proof. We recall the proof from [MV], Lemma 5.19. The requirement for E_i and E_{i+1} to have common ramification at a given $t \in \mathbb{P}^1 - \{z_1, \dots, z_s, \infty\}$ is at least "two conditions". Taking into account the one parameter variation of t, it is easy to see that $U - \mathrm{Fl}^o$ is of codimension at least one. Hence the inclusions $\mathrm{Fl}^o \subseteq U \subseteq \mathrm{Fl}$ are open and dense. □

Suppose $(V, E_\bullet) \in \mathrm{Fl}^o$. Then for $i = 1, \dots, N+1$, the polynomial $\mathrm{Wr}(E_i)$ is divisible by K_i and is of degree $i\,(d - i + 1) - \sum_{\ell=1}^{i} a_{N+1-i+\ell}(\infty)$. Introduce the polynomial y_i by the condition $\mathrm{Wr}(E_i) = K_i\, y_i$. The nonzero polynomial y_i is defined up to multiplication by a nonzero number. It has the following properties.

(α) If l_i is the degree of y_i, then

$$l_i = i\,(d - i + 1) - \sum_{\ell=1}^{i} a_{N+1-i+\ell}(\infty) - \sum_{s=1}^{n} \sum_{\ell=1}^{i} a_{N+1-i+\ell}(z_s). \qquad (2.3)$$

In particular, y_{N+1} is of degree 0.

(β) The polynomial y_i has no roots in the set $\{z_1, \dots, z_n\}$.

The fact that E_i and E_{i+1} have no common ramification points in $\mathbb{C} - \{z_1, \dots, z_n\}$ translates to the property

(γ) The polynomials y_i and y_{i+1} have no common roots.

Set $y_0 = 1$.

Suppose that $u_1, \dots, u_{N+1}$ is a basis of V such that for any $i = 1, \dots, N$, the elements $u_1, \dots, u_i$ form a basis of E_i. Let $Q_i = \mathrm{Wr}(u_1, \dots, u_{i-1}, u_{i+1}) \,/\, K_i$.

Lemma 2.5. $\mathrm{Wr}(y_i, Q_i) = T_i\, y_{i-1}\, y_{i+1}$ *for* $i = 1, \dots, N$.

Proof.

$$\begin{aligned} K_i^2\ \mathrm{Wr}(y_i, Q_i) &= \mathrm{Wr}(K_i\, y_i,\ K_i\, Q_i) \\ &= \mathrm{Wr}(\mathrm{Wr}(u_1, \dots, u_i), \mathrm{Wr}(u_1, \dots, u_{i-1}, u_{i+1})). \end{aligned}$$

By Lemma A.4 in [MV], the above quantity equals

$$\mathrm{Wr}(u_1, \dots, u_{i-1})\,\mathrm{Wr}(u_1, \dots, u_{i+1}) = K_{i-1}\, y_{i-1}\, K_{i+1}\, y_{i+1} = T_i\, y_i\, y_{i+1}\, {K_i}^2.$$

Finally, divide both sides by K_i^2. □

By Lemma 2.5, any multiple root of y_i is a root of either T_i, y_{i-1}, or y_{i+1}. Clearly we have

(δ) The polynomial y_i has no multiple roots.

(η) There exist $\tilde{y}_i \in \mathbb{C}[x]$ such that $\mathrm{Wr}(y_i, \tilde{y}_i) = T_i\, y_{i-1}\, y_{i+1}$, namely, $\tilde{y}_i = Q_i$.

We translate condition (η) into equations by using the following

Lemma 2.6. *Let $y \in \mathbb{C}[x]$ be a polynomial with no multiple roots and $T \in \mathbb{C}[x]$ any polynomial. Then equation $\mathrm{Wr}(y, \tilde{y}) = T$ has a solution $\tilde{y} \in \mathbb{C}[x]$ if and only if y divides $\mathrm{Wr}(y', T)$.*

This lemma follows from Lemma 6.1 below with $A = \mathbb{C}$.

Consider the space

$$R = \prod_{i=1}^{N} \mathbb{P}(\mathbb{C}_{l_i}[x])$$

where l_i is given by (2.3).

Let R^o be the open subset of R formed by the tuples $(y_1, \dots, y_N)$ satisfying conditions $(\alpha) - (\delta)$. Let $\mathcal{A}$ be the subset of R^o defined by the condition

$$y_i \text{ divides } \mathrm{Wr}(y_i',\ T_i\, y_{i-1} y_{i+1}) \text{ for } i = 1, \dots, N. \tag{2.4}$$

Using the monicity of y_i and long division, we can write the divisibility condition as a system of equations in the coefficients of y_i, y_{i-1}, and y_{i+1}. Hence $\mathcal{A}$ is a closed subscheme of R^o.

Consider the morphism

$$\Theta\, \mathrm{Fl}^o \longrightarrow R^o, \quad (V, E_\bullet) \longmapsto (y_1, \dots, y_N) = (\mathrm{Wr}(E_1)/K_1, \dots, \mathrm{Wr}(E_N)/K_N).$$

For $x \in \mathrm{Fl}^o$, condition (η) holds and by Lemma 6.1, Θ induces a morphism of schemes $\Theta\colon\ \mathrm{Fl}^o \to \mathcal{A}$.

Theorem 2.7. *The morphism $\Theta\colon\ \mathrm{Fl}^o \to \mathcal{A}$ is an isomorphism of schemes.*

It is proved in [MV] that Θ is a bijection of sets. In Section 3 we will extend the argument of [MV] to prove Theorem 2.7.

The following corollaries of Theorem 2.7 use the notion of the geometric multiplicity of an irreducible scheme. This notion, as well as its relation to intersection theory is reviewed in the appendix.

Corollary 2.8. *Let $x \in \Omega$. Let $m(x)$ be the length of the local ring of Ω at x. Let C be the irreducible component of $\mathcal{A}$ which, as a point set, is $\Theta(\pi^{-1}(x) \cap \mathrm{Fl}^o)$, see the Cartesian square (2.2). Then the geometric multiplicity of C equals $m(x)$. In particular, the geometric multiplicity of C equals the geometric multiplicity of Ω at x.*

Proof. Let Ω_x be the component of Ω containing x. As a set, Ω_x is just the point x. Consider the irreducible component $\mathcal{I}$ of Fl^o containing $\pi^{-1}(x)$,

$$\begin{array}{ccccc} \mathcal{I} & \longrightarrow & \mathrm{Fl} & \longrightarrow & \mathrm{Fl}(\mathcal{V}) \\ \downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow \\ \Omega_x & \longrightarrow & \Omega & \longrightarrow & \mathrm{Gr}(N+1, W) \end{array} \tag{2.5}$$

Since π is a locally trivial fiber bundle, the morphism $\mathcal{I} \to \Omega_x$ is a fiber bundle with smooth fibers for the Zariski topology on the scheme Ω_x. The multiplicity of $\mathcal{I}$ is the same as the multiplicity of its dense subset $\mathcal{I} \cap \mathrm{Fl}^o$. The corollary now follows from the theorem and Proposition 6.8. □

Corollary 2.9. *We have an equality of cohomology classes in* $H^*(\mathrm{Gr}(N+1, W))$,

$$[\Omega_{\boldsymbol{a}(\infty)}(\mathcal{F}(\infty))] \cdot \prod_{s=1}^{n} [\Omega_{\boldsymbol{a}(z_s)}(\mathcal{F}(z_s))] = c \cdot [\textit{class of a point}]$$

where c is the sum of the geometric multiplicities of irreducible components of $\mathcal{A}$.

2.4 Critical point equations

Consider the space $\tilde{R} = \prod_{i=1}^{N} \mathbb{C}^{l_i}$ with coordinates $(t_j^{(i)})$, where $i = 1, \dots, N$, $j = 1, \dots, l_i$. The product of symmetric groups $\Sigma = \Sigma_{l_1} \times \dots \times \Sigma_{l_N}$ acts on $\tilde{R}$ by permuting coordinates with the same upper index. Define a map

$$\Gamma \colon \tilde{R} \longrightarrow R, \quad (t_j^{(i)}) \longmapsto (y_1, \dots, y_N),$$

where $y_i = \prod_{j=1}^{l_i}(x - t_j^{(i)})$. Define the scheme $\tilde{\mathcal{A}}$ by the condition $\tilde{\mathcal{A}} = \Gamma^{-1}(\mathcal{A})$. The natural map $\tilde{\mathcal{A}} \to \mathcal{A}$ is finite and étale. The scheme $\tilde{\mathcal{A}}$ is Σ-invariant. The scheme $\tilde{\mathcal{A}}$ lies in the Σ-invariant subspace $\tilde{R}^o$ of all $(t_j^{(i)})$ with the following properties for every i:

- the numbers $t_1^{(i)}, \dots, t_{l_i}^{(i)}$ are distinct;
- the sets $\{t_1^{(i)}, \dots, t_{l_i}^{(i)}\}$ and $\{t_1^{(i+1)}, \dots, t_{l_{i+1}}^{(i+1)}\}$ do not intersect;
- the sets $\{t_1^{(i)}, \dots, t_{l_i}^{(i)}\}$ and $\{z_1, \dots, z_n\}$ do not intersect.

Lemma 2.10.

- *The connected components of $\mathcal{A}$ and $\tilde{\mathcal{A}}$ are irreducible.*
- *The reduced schemes underlying $\mathcal{A}$ and $\tilde{\mathcal{A}}$ are smooth.*
- *If C is a connected component of $\mathcal{A}$, then the group Σ acts transitively on the connected components of* $\Gamma^{-1}(C)$.

Proof. By Theorem 2.7, $\mathcal{A}$ is isomorphic to Fl^o. The reduced scheme underlying Fl^o is smooth. Therefore the reduced scheme underlying $\mathcal{A}$ is smooth. Since Γ is étale, we deduce that the reduced scheme underlying $\tilde{\mathcal{A}}$ is also smooth.

The smoothness conclusions immediately imply the irreducibility of connected components of $\mathcal{A}$ and $\tilde{\mathcal{A}}$.

The transitivity assertion follows from the fact that Γ is a Galois covering with Galois group Σ. □

Consider on $\tilde{R}^o$ the regular rational function

$$\Phi(t_j^{(i)}) = \prod_{i=1}^{N}\prod_{j=1}^{l_i} T_i(t_j^{(i)})^{-1} \prod_{i=1}^{N-1}\prod_{j=1}^{l_i}\prod_{k=1}^{l_{i+1}}(t_j^{(i)} - t_k^{(i+1)})^{-1} \prod_{i=1}^{N}\prod_{1\le j<k\le l_i}(t_j^{(i)} - t_k^{(i)})^2.$$

This Σ-invariant function is called *the master function associated with the basic situation.*

Define the scheme $\tilde{\mathcal{A}}'$ as the subscheme in $\tilde{R}^o$ of critical points of the master function.

Lemma 2.11. *The subschemes $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{A}}'$ of $\tilde{R}^o$ coincide.*

Proof. The subscheme $\tilde{\mathcal{A}}$ is defined by divisibility conditions (2.4). By Lemma 6.4, the divisibility condition (2.4) for a fixed i, reduces to the critical point equations for the function

$$\prod_{j=1}^{l_i} T_i(t_j^{(i)})^{-1} \prod_{j=1}^{l_i}\prod_{k=1}^{l_{i+1}}(t_j^{(i)} - t_k^{(i+1)})^{-1} \prod_{j=1}^{l_i}\prod_{k=1}^{l_{i-1}}(t_k^{(i-1)} - t_j^{(i)})^{-1} \prod_{1\le j<k\le l_i}(t_j^{(i)} - t_k^{(i)})^2$$

of variables $t_1^{(i)}, \dots, t_{l_i}^{(i)}$. Then the system of divisibility conditions (2.4) for all i together is just the critical scheme of Φ. This concludes the proof. □

Let $(V, E_\bullet) \in \mathrm{Fl}^o$. Denote $\boldsymbol{y} = \Theta(V, E_\bullet) \in \mathcal{A}$. Pick a point $\boldsymbol{t} \in \Gamma^{-1}(\boldsymbol{y})$. Let C be the unique irreducible component of $\mathcal{A}$ containing $\boldsymbol{y}$ and $\tilde{C}$ the unique irreducible component of $\tilde{\mathcal{A}}$ containing $\boldsymbol{t}$.

Theorem 2.12. *The geometric multiplicity of the scheme Ω at V equals the geometric multiplicity of $\tilde{C}$.*

Proof. The morphism $\tilde{C} \to C$ is étale. By Proposition 6.8, the geometric multiplicity of $\tilde{C}$ coincides with that of C. Now the theorem follows from Corollary 2.8. □

The group Σ acts on the set of connected components of $\tilde{\mathcal{A}}$. For an orbit of this action, define its geometric multiplicity to be the geometric multiplicity of any member of the orbit.

Corollary 2.13. *We have an equality of cohomology classes in* $H^*(\mathrm{Gr}(N+1, W))$,

$$[\Omega_{\boldsymbol{a}(\infty)}(\mathcal{F}(\infty))] \cdot \prod_{s=1}^{n} [\Omega_{\boldsymbol{a}(z_s)}(\mathcal{F}(z_s))] = c \cdot [\textit{class of a point}]$$

where c *is the number of orbits for the action of* Σ *on the connected components of* $\tilde{\mathcal{A}}$ *counted with geometric multiplicity.*

3 Proof of Theorem 2.7

3.1 Admissible modules

Let $(A, \mathfrak{m})$ be a local ring with residue field $\mathbb{C}$. A submodule $V \subset A[x]$ is said to be *an admissible submodule of rank* $N+1$, if

- the submodule $V \subset A[x]$ is a free A-module of rank $N+1$;
- the morphism $V \otimes (A/\mathfrak{m}) \to (A/\mathfrak{m})[x] = \mathbb{C}[x]$ is injective.

An admissible submodule $V \subset A[x]$ is said to have *ramification sequence* $\boldsymbol{a}(z) = (a_1, \dots, a_{N+1})$ at $z \in \mathbb{C}$ if V has an A-basis

$$(x-z)^{N+1-1+a_1} f_1,\ (x-z)^{N+1-2+a_2} f_2, \dots,\ (x-z)^{N+1-(N+1)+a_{N+1}} f_{N+1}$$

with $f_i(z) \in A^*$ for $i = 1, \dots, N+1$.

An admissible submodule $V \subset A[x]$ is said to have *ramification sequence* $\boldsymbol{a}(\infty) = (a_1, \dots, a_{N+1})$ at ∞ if V has an A-basis $f_1, f_2, \dots, f_{N+1}$ with monic f_i and $\deg f_i = d - (N+1) + i - a_i$ for $i = 1, \dots, N+1$.

Remark 3.1. Admissible modules $V \subset A[x]$ of rank $N+1$ are in one-to-one correspondence with morphisms $\mathrm{Spec}(A) \to \mathrm{Gr}(N+1, \mathbb{C}[x])$. Intuitively, if $A = \mathbb{C}[[t]]$, this is a formal holomorphic map of a 1-disc into $\mathrm{Gr}(N+1, \mathbb{C}[x])$.

An admissible submodule $V \subset A[x]$ may not have a ramification sequence at a given $z \in \mathbb{C} \cup \{\infty\}$. This corresponds intuitively to the case when the formal map considered above does not remain in a Schubert cell. If V has a ramification sequence at z, then the ramification sequence is unique (equal to the ramification sequence of the subspace $V \otimes (A/\mathfrak{m}) \subset \mathbb{C}[x]$ at z).

For $f \in A[x]$, we denote by $\bar{f}$ the corresponding polynomial in $(A/\mathfrak{m})[x] = \mathbb{C}[x]$. The following standard lemma is proved in Section 6.2.

Lemma 3.2. *Let* A *be a local ring with residue field* $\mathbb{C}$ *and* $V \subset A[x]$ *a submodule. Then the following statements are equivalent.*

(1) *The submodule* V *is admissible.*

(2) *The submodule V is a finitely generated A-module, and there is an A-module decomposition $A[x] = V \oplus M$ for some A-module M.*

(3) *For some k, there exist $u_1, \dots, u_k \in V$ such that V is the A-span of $u_1, \dots, u_k$ and the elements $\bar{u}_1, \dots, \bar{u}_k \in \mathbb{C}[x]$ are linearly independent over $\mathbb{C}$.*

Lemma 3.3. *Let A be a local ring with residue field $\mathbb{C}$ and $V \subset A[x]$ an admissible submodule of rank $N+1$. Let $u_1, \dots, u_{N+1}$ be a basis of V as an A-module. Suppose $v \in A[x]$ satisfies the equation $\mathrm{Wr}(u_1, \dots, u_{N+1}, v) = 0$. Then $v \in V$.*

Lemma 3.3 is proved in Section 3.3.

3.2 Proof of Theorem 2.7

Our first objective will be to show that $\Theta\colon \mathrm{Fl}^o \to R^o$ is a closed embedding of schemes. Our second objective will be to show that $\mathrm{Fl}^o \to \mathcal{A}$ is an isomorphism of schemes.

Clearly the morphism $\Theta\colon \mathrm{Fl}^o \to R^o$ extends to a morphism of projective schemes $\tilde{\Theta}\colon \mathrm{Fl} \to R$ and $\mathrm{Fl}^o = \tilde{\Theta}^{-1}(R^o)$. The morphism $\tilde{\Theta}$ is closed as a morphism of projective schemes.

We achieve the first objective by showing that $\tilde{\Theta}$ is an embedding. By Lemma 6.9, it will be enough to show that for every local ring A over $\mathbb{C}$, the morphism $\mathrm{Fl}(A) \to R(A)$ is an injective map of sets.

To achieve the second objective, by Lemma 6.10, it will be enough to show that for any local ring A over $\mathbb{C}$, the induced map $\mathrm{Fl}^o(A) \to \mathcal{A}(A)$ is a set theoretic surjection.

To use these criteria, we need to define the sets $\mathrm{Fl}(A)$, $\mathrm{Fl}^o(A)$, $R(A)$, and $\mathcal{A}(A)$ more explicitly.

By Proposition 6.11, the set $\mathrm{Fl}(A)$ is the set of pairs $(V, E_\bullet)$, such that

- $V \subset A[x]$ is an admissible submodule of rank $N+1$;
- $E_\bullet = (E_1 \subsetneq E_2 \subsetneq \cdots \subsetneq E_{N+1} = V)$ is a filtration by admissible submodules;
- V has the ramification sequence $(a_1(z), \dots, a_{N+1}(z))$ at each $z \in \{z_1, \dots, z_n, \infty\}$.

The subset $\mathrm{Fl}^o(A)$ is the set of $(V, E_\bullet) \in \mathrm{Fl}(A)$ such that the induced point in $\mathrm{Fl}(\mathbb{C})$ is a point of Fl^o.

The set $R(A)$ is the set

$$\{(y_1, \dots, y_N) \in \prod_{i=1}^{N} A_{l_i}[x] \mid \bar{y}_i \neq 0,\ i = 1, \dots, N\}$$

modulo the equivalence relation $(y_1, \dots, y_N) \sim (\tilde{y}_1, \dots, \tilde{y}_N)$ if there exist $a_i \in A^*$ such that $a_i\, y_i = \tilde{y}_i$ for $i = 1, \dots, N$.

The subset $\mathcal{A}(A)$ consists of elements $(y_1, \dots, y_N) \in R(A)$ such that

- For $i = 1, \dots, N$, condition (η) holds. By Lemma 6.1, this implies that the polynomial y_i divides $\mathrm{Wr}(y_i', T_i\, y_{i-1}\, y_{i+1})$. Here $y_0 = y_{N+1} = 1$.
- The reduction $(\bar{y}_1, \dots, \bar{y}_N) \in R(\mathbb{C})$ is a point of R^o.

Lemma 3.4. *The morphism* $\tilde{\Theta}\colon \mathrm{Fl} \to R$ *is a closed embedding of schemes.*

Proof. We need to show that $\tilde{\Theta}\colon \mathrm{Fl}(A) \to R(A)$ is a set theoretic injection for any local ring A.

Suppose $(V, E_\bullet)$ and $(V', E'_\bullet)$ are two points of $\mathrm{Fl}(A)$ with $\tilde{\Theta}(V, E_\bullet) = \tilde{\Theta}(V', E'_\bullet)$. Pick bases $(u_1, \dots, u_{N+1})$ and $(v_1, \dots, v_{N+1})$ for V and V' respectively so that for all i, the admissible submodule E_i is the A-span of $u_1, \dots, u_i$ and the admissible submodule E_i' is the A-span of $v_1, \dots, v_i$.

The hypothesis implies that $\mathrm{Wr}(u_1, \dots, u_i) = c_i\, \mathrm{Wr}(v_1, \dots, v_i)$ with $c_i \in A^*$. Clearly $E_1 = E_1'$. Assume by induction that $E_i = E_i'$. Then

$$\mathrm{Wr}(v_1, \dots, v_{i+1}) = c\, \mathrm{Wr}(u_1, \dots, u_i, v_{i+1}) = c'\, \mathrm{Wr}(u_1, \dots, u_i, u_{i+1})$$

for $c, c' \in A^*$. Therefore

$$\mathrm{Wr}(u_1, \dots, u_i, cv_{i+1} - c'u_{i+1}) = 0.$$

Lemma 3.3 implies $cv_{i+1} - c'u_{i+1} \in E_i$ and hence $E_{i+1} = E'_{i+1}$. □

We need to show that for any local ring A over $\mathbb{C}$, the induced map $\mathrm{Fl}^o(A) \to \mathcal{A}(A)$ is a set theoretic surjection. But this claim on surjectivity follows from the following theorem on the existence of solutions to Wronskian equations.

Let $T_0, T_1, \dots, T_N \in \mathbb{C}[x] \subseteq A[x]$ be non-zero polynomials. Let $S \subset \mathbb{C}$ be the union of their zero sets. Set $K_i = T_0^i\, T_1^{i-1}\, T_2^{i-2} \cdots T_{i-1}$ for $i = 1, \dots, N+1$.

Let $y_1, \dots, y_N \in A[x]$ be monic polynomials of arbitrary degree. Set $y_0 = y_{N+1} = 1$.

For $z \in S$ and $i = 1, \dots, N+1$, define

$$e_i(z) = i - 1 + \sum_{j=0}^{i-1} \mathrm{ord}_z\, T_j, \quad c_i = i - 1 + \deg y_i - \deg y_{i-1} + \sum_{j=0}^{i-1} \deg T_j.$$

Theorem 3.5. *Under these conditions assume that for all i*

- *the polynomial* $\bar{y}_i \in \mathbb{C}[x]$ *has no multiple roots, no roots in S, and is coprime to* $\bar{y}_{i+1}$;
- *the polynomial* y_i *divides* $\mathrm{Wr}(y_i', T_i\, y_{i-1}\, y_{i+1})$.

Then there exist $u_1, \dots, u_{N+1} \in A[x]$ *with the following properties. Set* E_i *to be the A-span of* $u_1, \dots, u_i$ *for* $i = 1, \dots, N+1$ *and set* $V = E_{N+1}$. *Then for all i*

(1) $\mathrm{Wr}(u_1, \dots, u_i) = K_i\, y_i$;

(2) $E_i \subset A[x]$ *is admissible*;

(3) E_i *has ramification sequences at each* $z \in S \cup \{\infty\}$. *The set of exponents at* $z \in S$ *is* $\{e_1(z), \dots, e_i(z)\}$. *The set of exponents at* ∞ *is* $\{c_1, \dots, c_i\}$.

Theorem 3.5 is proved in Section 3.4.

Let $(y_1, \dots, y_N) \in \mathcal{A}(A)$. Apply Theorem 3.5 and obtain a point $(V, E_\bullet)$. It is easy to see that $(V, E_\bullet) \in \mathrm{Fl}^o(A)$ and $\Theta(V, E_\bullet) = (y_1, \dots, y_N)$. The proof of Theorem 2.7 is complete.

3.3 Proof of Lemma 3.3

Let $\mathfrak{m} \subset A$ be the maximal ideal. Let $u_1, \dots, u_{N+1} \in V$ be a basis, $u_i = \sum_l a_l^i x^l$ with $a_i^l \in A$.

If necessary changing the basis, we may assume the basis has the following property. There exist nonnegative integers $k_1 > \cdots > k_{N+1}$ such that

$$a_{k_i}^j = 0 \text{ for } j \neq i\,; \quad a_{k_i}^i \in A - m\,; \quad a_j^i \in m \text{ for } j > a_{k_i}^i .$$

Let $v = \sum_l b_l x^l$ be a nonzero polynomial such that $\mathrm{Wr}(u_1, \dots, u_{N+1}, v) = 0$. We may assume that $b_{k_i} = 0$ for all i. We shall prove that this leads to contradiction.

Recall that for any nonzero $a \in A$, there is a unique r such that $a \in m^r - m^{r+1}$, see Krull's intersection theorem ([M], Theorem 8.10).

Let r be the smallest number such that some b_l is in $m^r - m^{r+1}$, and p the largest index such that $b_p \in m^r - m^{r+1}$. Clearly $p \notin \{k_1, \dots, k_{N+1}\}$. The polynomial $\mathrm{Wr}(u_1, \dots, u_{N+1}, v)$ is nonzero since in its decomposition into monomials, the coefficient of the monomial $\mathrm{Wr}(x^{k_1}, \dots, x^{k_{N+1}}, x^p)$ belongs to $m^r - m^{r+1}$.

3.4 Proof of Theorem 3.5

The proof follows [MV]. Call a tuple $(y_1, \dots, y_N)$ *fertile* if it satisfies the conditions of Theorem 3.5.

For $i = 1, \dots, N$, define the process of reproduction in the i-th direction. Namely, find a solution $\tilde{y}_i \in A[x]$ to the equation $\mathrm{Wr}(y_i, \tilde{y}_i) = T_i\, y_{i+1}\, y_{i-1}$. If $\tilde{y}_i$ is a solution, then for $c \in A$, the polynomial $\tilde{y}_i + c y_i$ is a solution too. Add to $\tilde{y}_i$ the term $c y_i$ if necessary, and obtain a monic $\tilde{y}_i$ such that its reduction modulo the maximal ideal does not have roots in S, does not have multiple roots, and has no common roots with reductions of y_{i-1} or y_{i+1}. The transformation from the tuple $(y_1, \dots, y_n)$ to $(y_1, \dots, y_{i-1}, \tilde{y}_i, y_{i+1}, \dots, y_N)$ is the process of reproduction in the i-th direction.

Claim. *The tuple* $(y_1, \dots, \tilde{y}_i, \dots, y_N)$ *is fertile.*

To prove the claim it is enough to show that

$$y_{i-1} \text{ divides } \mathrm{Wr}(y'_{i-1}, T_{i-1}\, y_{i-2}\, \tilde{y}_i) \tag{3.1}$$

and

$$y_{i+1} \text{ divides } \mathrm{Wr}(y'_{i+1},\ T_{i+1}\, y_{i+2}\, \tilde{y}_i). \tag{3.2}$$

We prove (3.1). Statement (3.2) is proved similarly.

Clearly, y_{i-1} divides $\mathrm{Wr}(y_i, \tilde{y}_i)$, and y_{i-1} divides $\mathrm{Wr}(y'_{i-1},\ T_{i-1}\, y_{i-2}\, y_i)$ by assumption. By Jacobi's rule,

$$\mathrm{Wr}(y'_{i-1},\ T_{i-1}\, y_{i-2}\, \tilde{y}_i)\, y_i - \mathrm{Wr}(y'_{i-1},\ T_{i-1}\, y_{i-2}\, y_i)\, \tilde{y}_i = \mathrm{Wr}(y_i, \tilde{y}_i)\, T_{i-1}\, y_{i-2}\, y'_{i-1}.$$

Hence y_{i-1} divides $\mathrm{Wr}(y'_{i-1},\ T_{i-1}\, y_{i-2}\, \tilde{y}_i)\, y_i$. The ideal $(\bar{y}_{i-1}, \bar{y}_i)$ equals $\mathbb{C}[x]$, by assumption. By Lemma 6.3, this implies that the ideal (y_{i-1}, y_i) equals $A[x]$. Now use Lemma 6.5 to see that y_{i-1} divides $\mathrm{Wr}(y'_{i-1},\ T_{i-1}\, y_{i-2}\, \tilde{y}_i)$. This proves (3.1) and the claim.

To construct the polynomials $u_1, \dots, u_{N+1}$ we do the following. We set $u_1 = K_1\, y_1$. To construct the polynomial u_{i+1}, for $i = 1, \dots, N$, we perform the simple reproduction procedure in the i-th direction and obtain the tuple $(y_1, \dots, \tilde{y}_i, \dots, y_N)$. Next we perform for $(y_1, \dots, \tilde{y}_i, \dots, y_N)$ the simple reproduction procedure in the direction of $i - 1$ and obtain $(y_1, \dots, \tilde{y}_{i-1}, \tilde{y}_i, \dots, y_N)$. We repeat this procedure all the way until the simple reproduction procedure in the first direction and obtain $(\tilde{y}_1, \dots, \tilde{y}_{i-1}, \tilde{y}_i, \dots, y_N)$. We set $u_{i+1} = K_1 \tilde{y}_1$.

Claim. *We have* $\mathrm{Wr}(u_1, \dots, u_i) = K_i\, y_i$ *for* $i = 2, \dots, N + 1$.

We prove the claim by induction. Let $(y_1, \dots, \tilde{y}_i, \dots, y_N)$ be the tuple obtained by the simple reproduction in the i-th direction. Apply induction to the tuple $(y_1, \dots, \tilde{y}_i, \dots, y_N)$ to obtain $\mathrm{Wr}(u_1, \dots, u_{i-1}, u_{i+1}) = K_i\, \tilde{y}_i$. Induction applied to the tuple $(y_1, \dots, y_i, \dots, y_N)$ gives $\mathrm{Wr}(u_1, \dots, u_{i-1}, u_i) = K_i y_i$. Now

$$\begin{aligned}\mathrm{Wr}(u_1, \dots, u_{i+1})\, \mathrm{Wr}(u_1, \dots, u_{i-1}) &= \mathrm{Wr}(\mathrm{Wr}(u_1, \dots, u_{i-1}, u_i),\\ &\quad \mathrm{Wr}(u_1, \dots, u_{i-1}, u_{i+1})) = \mathrm{Wr}(K_i\, y_i,\ K_i\, \tilde{y}_i) = K_i^2\, T_i\, y_{i_1}\, y_{i+1}.\end{aligned}$$

By induction we also have

$$\mathrm{Wr}(u_1, \dots, u_{i-1}) = K_{i-1}\, y_{i-1} \quad \text{and} \quad T_i = K_{i+1}\, K_{i-1}/K_i^2.$$

The above equation rearranges to

$$\mathrm{Wr}(u_1, \dots, u_{i+1})\, K_{i-1}\, y_{i-1} = K_{i+1}\, K_{i-1}\, y_{i-1}\, y_{i+1},$$

which gives the desired equality $\mathrm{Wr}(u_1, \dots, u_{i+1}) = K_{i+1}\, y_{i+1}$.

Return to the proof of Theorem 3.5. Let E_i be the A-span of $u_1, \dots, u_i$. Then $\mathrm{Wr}(\bar{u}_1, \dots, \bar{u}_i) = \overline{K}_i\, \bar{y}_i = K_i\, \bar{y}_i \neq 0$. By part (3) of Lemma 3.2, the submodule $E_i \subset A[x]$ is admissible. This proves (1) and (2) of the theorem.

Now we will calculate the exponents of E_i at $z \in S$. The exponents of E_i at ∞ are calculated similarly.

By induction assume that the set of exponents of E_i at z is $\{e_1(z), \dots, e_i(z)\}$. That means that E_i has an A-basis $v_1, \dots, v_i$ such that $v_j = (x-z)^{e_j(z)} a_j$ with $a_j(z) \in A^*$ for all j. Hence $\mathrm{Wr}(E_i)$ is of the form $(x-z)^b a$ with

$$b = \sum_{\ell=1}^{i} e_\ell(z) - \frac{i\,(i-1)}{2}, \quad a(z) \in A^*.$$

We have $b = \mathrm{ord}_z\ K_i$ since $\mathrm{Wr}(E_i) = K_i\, y_i$.

Let $v_{i+1} \in E_{i+1}$ be such that $v_1, \dots, v_i, v_{i+1}$ is an A-basis of E_{i+1}. We may assume that $v_{i+1} = \sum_{\ell=0}^{d} c_\ell\ (x-z)^\ell$ with coefficients c_ℓ equal to zero for $\ell \in \{e_1(z), \dots, e_i(z)\}$. Let p be the smallest integer with $c_p \neq 0$. It is easy to see that the Wronskian of E_{i+1} equals $(x-z)^{b+p-i} h$ where $h(z)$ is c_p up to multiplication by an element in A^*. From equation $\mathrm{Wr}(E_{i+1}) = K_{i+1}\, y_{i+1}$ we deduce that $\mathrm{Wr}(E_{i+1})$ is of the form $(x-z)^{\mathrm{ord}_z K_{i+1}} g$ with $g(z) \in A^*$. Hence $b + p - i = \mathrm{ord}_z K_{i+1}$, $p = e_{i+1}(z)$, and $c_p \in A^*$.

This argument shows that $e_{i+1}(z)$ does not belong to the set $\{e_1(z), \dots, e_i(z)\}$. Therefore E_{i+1} has a ramification sequence at z and the set of exponents at z is $\{e_1(z), \dots, e_i(z), e_{i+1}(z)\}$. The induction step is complete.

4 Schubert cells and critical points

Let Σ_{N+1} be the permutation group of the set $\{1, \dots, N+1\}$ and $w \in \Sigma_{N+1}$. Assume that a basic situation of Section 2.3 is given. In Section 2.3, we defined a flag bundle Fl over Ω. We also observed that $V \in \Omega$ has $n+1$ distinguished complete flags on V induced from the complete flags $\mathcal{F}(z_1), \dots, \mathcal{F}(z_n)$ and $\mathcal{F}(\infty)$ of W.

Let us write the set $\{d - (N+1) + j - a_j(\infty) \mid j = 1, \dots, N+1\}$ of exponents of V at ∞ as $\{c_1, \dots, c_{N+1}\}$, where $c_1 > \dots > c_{N+1}$.

We define a subset $\mathrm{Fl}_w^o \subset \mathrm{Fl}$ as follows. Let U_w be the subset of Fl formed by points $(V, E_\bullet)$ such that

- $E_\bullet \in \mathrm{Fl}(V)$ lies in the intersection of n open Schubert cells corresponding respectively to the n distinguished complete flags on V induced from the flags $\mathcal{F}(z_1), \dots, \mathcal{F}(z_n)$. This condition on $E_\bullet$ is equivalent to the statement that for $i = 1, \dots, N+1$ and $z \in \{z_1, \dots, z_n\}$, the subspace E_i has ramification sequence $(a_{N+1-i+1}(z), \dots, a_{N+1}(z))$.
- $E_\bullet \in \mathrm{Fl}(V)$ lies in the Schubert cell, corresponding to the permutation w and the distinguished complete flag on V induced from the flag $\mathcal{F}(\infty)$. This condition on $E_\bullet$ is equivalent to the statement that the set of exponents of E_i at ∞ is $\{c_{w(1)}, \dots, c_{w(i)}\}$.

We define $\mathrm{Fl}_w^o \subseteq U_w$ as the subset of points $(V, E_\bullet)$ such that for all i, the subspaces $E_i \subset \mathbb{C}[x]$ and $E_{i+1} \subset \mathbb{C}[x]$ do not have common ramification points in $\mathbb{C} - \{z_1, \dots, z_n\}$.

Notice that the subset Fl^o_w may be empty if w is not the identity element in Σ_{N+1}.

Lemma 4.1. *The morphism* $\mathrm{Fl}^o_w \to \Omega$ *is smooth.*

Proof. Let $\mathcal{J}$ be the subset of Fl formed by points $(V, E_\bullet)$ such that $E_\bullet$ lies in the Schubert cell corresponding to the permutation w and the distinguished flag on V induced from $\mathcal{F}(\infty)$. It is easy to see that U_w is an open subset of $\mathcal{J}$ and therefore it suffices to show that $\mathcal{J} \to \Omega$ is smooth.

Denote by $\mathcal{V}_{\boldsymbol{a}(\infty)}$ the pull-back of $\mathcal{V}$ to $\Omega^o_{\boldsymbol{a}(\infty)}(\mathcal{F}(\infty)) \hookrightarrow \mathrm{Gr}(N+1, W)$. There is a distinguished section of $\mathrm{Fl}(\mathcal{V}_{\boldsymbol{a}(\infty)}) \to \Omega^o_{\boldsymbol{a}(\infty)}(\mathcal{F}(\infty))$, see Section 2.1. Let $G^{\mathcal{F}(\infty)}_w \subset \mathrm{Fl}(\mathcal{V}_{\boldsymbol{a}(\infty)})$ be the part corresponding to w in the partition of $\mathrm{Fl}(\mathcal{V}_{\boldsymbol{a}(\infty)})$ into Schubert cells associated to this distinguished section.

There is a fiber square

$$\begin{array}{ccc} \mathcal{J} & \longrightarrow & G^{\mathcal{F}(\infty)}_w \\ \Big\downarrow & & \Big\downarrow p \\ \Omega & \longrightarrow & \Omega^o_{\boldsymbol{a}(\infty)}(\mathcal{F}(\infty)) \end{array}$$

The morphism p is clearly smooth. The morphism $\mathcal{J} \to \Omega$ is the base change of a smooth morphism and hence it is smooth too. $\square$

Let $(V, E_\bullet) \in \mathrm{Fl}^o_w$. For $i = 1, \dots, N+1$, the polynomial $\mathrm{Wr}(E_i)$ is divisible by K_i and has degree $\sum_{j=1}^{i} c_{w(j)} - i(i-1)/2$. Introduce the polynomial y_i by the condition $\mathrm{Wr}(E_i) = K_i\, y_i$. Then:

- $(\alpha)_w$ If l^w_i is the degree of y_i, then $l^w_i = \sum_{j=1}^{i} c_{w(j)} - i(i+1)/2 - \deg K_i$. In particular, y_{N+1} is of degree 0.
- $(\beta)_w$ The polynomial y_i has no roots in the set $\{z_1, \dots, z_n\}$.
- $(\gamma)_w$ The polynomials y_i and y_{i+1} have no common roots.
- $(\delta)_w$ The polynomial y_i has no multiple roots.
- $(\eta)_w$ There exist $\tilde{y}_i \in \mathbb{C}[x]$ such that $\mathrm{Wr}(y_i, \tilde{y}_i) = T_i\, y_{i-1}\, y_{i+1}$.

Consider the space

$$R_w = \prod_{i=1}^{N} \mathbb{P}(\mathbb{C}_{l^w_i}[x])$$

where l^w_i is given by property $(\alpha)_w$.

Let R^o_w be the open subset of R_w formed by the tuples $(y_1, \dots, y_N)$ satisfying conditions $(\alpha)_w - (\delta)_w$. Let $\mathcal{A}_w$ be the subset of R^o defined by the condition

$$y_i \text{ divides } \mathrm{Wr}(y_i',\ T_i\, y_{i-1} y_{i+1}) \text{ for } i = 1, \dots, N, \qquad (4.1)$$

Using the monicity of y_i and long division, we can write the divisibility condition as a system of equations in the coefficients of y_i, y_{i-1}, and y_{i+1}. Hence $\mathcal{A}_w$ is a closed subscheme of R^o_w.

Consider the morphism

$$\Theta_w\colon\ \mathrm{Fl}^o_w \longrightarrow R^o_w, \quad (V, E_\bullet) \longmapsto (y_1, \dots, y_N) = (\mathrm{Wr}(E_1)/K_1, \dots, \mathrm{Wr}(E_N)/K_N).$$

For $x \in \mathrm{Fl}^o_w$, condition $(\eta)_w$ holds and by Lemma 2.6, Θ induces a morphism of schemes $\Theta\colon\ \mathrm{Fl}^o_w \to \mathcal{A}_w$.

Theorem 4.2. *The morphism* $\Theta_w\colon\ \mathrm{Fl}^o_w \to \mathcal{A}_w$ *is an isomorphism of schemes.*

The proof of Theorem 4.2 is similar to that of Theorem 2.7. Notice that in Theorem 3.5, no assumptions were made on the degrees of $y_1, \dots, y_N$.

Consider the space $\tilde{R}_w = \prod_{i=1}^N \mathbb{C}^{l_i^w}$ with coordinates $(t_j^{(i)})$, where $i = 1, \dots, N$, $j = 1, \dots, l_i^w$. The product of symmetric groups $\Sigma^w = \Sigma_{l_1^w} \times \cdots \times \Sigma_{l_N^w}$ acts on $\tilde{R}$ by permuting coordinates with the same upper index. Define a map

$$\Gamma\colon\ \tilde{R}_w \longrightarrow R, \quad (t_j^{(i)}) \longmapsto (y_1, \dots, y_N),$$

where $y_i = \prod_{j=1}^{l_i^w} (x - t_j^{(i)})$. Define the scheme $\tilde{\mathcal{A}}_w$ by the condition $\tilde{\mathcal{A}}_w = \Gamma^{-1}(\mathcal{A}_w)$. The natural map $\tilde{\mathcal{A}}_w \to \mathcal{A}$ is finite and étale. The scheme $\tilde{\mathcal{A}}_w$ is Σ^w-invariant. The scheme $\tilde{\mathcal{A}}_w$ lies in the Σ^w-invariant subspace $\tilde{R}^o$ of all $(t_j^{(i)})$ with the following properties for every i:

- the numbers $t_1^{(i)}, \dots, t_{l_i^w}^{(i)}$ are distinct;
- the sets $\{t_1^{(i)}, \dots, t_{l_i^w}^{(i)}\}$ and $\{t_1^{(i+1)}, \dots, t_{l_{i+1}^w}^{(i+1)}\}$ do not intersect;
- the sets $\{t_1^{(i)}, \dots, t_{l_i^w}^{(i)}\}$ and $\{z_1, \dots, z_n\}$ do not intersect.

The following lemma is proved in the same manner as Lemma 2.10.

Lemma 4.3.

- *The connected components of* $\mathcal{A}_w$ *and* $\tilde{\mathcal{A}}_w$ *are irreducible.*
- *The reduced schemes underlying* $\mathcal{A}_w$ *and* $\tilde{\mathcal{A}}_w$ *are smooth.*
- *If* C *is a connected component of* $\mathcal{A}_w$*, then the group* Σ^w *acts transitively on the connected components of* $\Gamma^{-1}(C)$.

Consider on $\tilde{R}^o_w$ the regular rational function

$$\Phi_w(t_j^{(i)}) = \prod_{i=1}^{N}\prod_{j=1}^{l_i^w} T_i(t_j^{(i)})^{-1} \prod_{i=1}^{N-1}\prod_{j=1}^{l_i^w}\prod_{k=1}^{l_{i+1}^w}(t_j^{(i)} - t_k^{(i+1)})^{-1} \prod_{i=1}^{N}\prod_{1\le j<k\le l_i^w}(t_j^{(i)} - t_k^{(i)})^2.$$

This Σ^w-invariant function is called *the master function associated with the basic situation and the permutation* $w \in \Sigma_{N+1}$.

Define the scheme $\tilde{\mathcal{A}}'_w$ as the subscheme in $\tilde{R}^o$ of critical points of the master function. The following is a generalization of Lemma 2.11 and is proved in an identical fashion.

Lemma 4.4. *The subschemes $\tilde{\mathcal{A}}_w$ and $\tilde{\mathcal{A}}'_w$ of $\tilde{R}^o$ coincide.*

Let $(V, E_\bullet) \in \mathrm{Fl}^o_w$. Denote $\boldsymbol{y} = \Theta(V, E_\bullet) \in \mathcal{A}_w$. Pick a point $\boldsymbol{t} \in \Gamma^{-1}(\boldsymbol{y})$. Let C be the unique irreducible component of $\mathcal{A}_w$ containing $\boldsymbol{y}$ and $\tilde{C}$ the unique irreducible component of $\tilde{\mathcal{A}}_w$ containing $\boldsymbol{t}$. The following is a generalization of Theorem 2.12 with a similar proof.

Theorem 4.5. *The geometric multiplicity of the scheme Ω at V equals the geometric multiplicity of $\tilde{C}$.*

Example. Let $N = 1$, $n = 3$, $z_1 = 1$, $z_2 = \omega$, $z_3 = \omega^2$ where $\omega = e^{\frac{2\pi i}{3}}$. Let $d = 3$, $\boldsymbol{a}(1) = \boldsymbol{a}(\omega) = \boldsymbol{a}(\omega^2) = \boldsymbol{a}(\infty) = (1, 0)$. Let w be the transposition (12) in Σ^2.

It is easy to see that $T_1(x) = x^3 - 1$, $l_1^w = 1$, and $\Phi(t) = T_1(t)^{-1}$. The critical scheme of Φ is $\{t \mid t^2 = 0\}$, namely $\mathrm{Spec}(\mathbb{C}[t]/(t^2))$. In other words, the master function has one critical point at $t = 0$ of multiplicity 2.

The polynomial y_1 associated to the critical point is the polynomial x. The equation $\mathrm{Wr}(y_1, \tilde{y_1}) = T_1$ has solutions $\tilde{y_1} = 1 + x^3/2 - cx$ with $c \in \mathbb{C}$. The associated 2-dimensional space of polynomials V is the $\mathbb{C}$-span of x and $x^3 + 2$. The ramification points of V are $1, \omega, \omega^2, \infty$ with ramification sequences all equal to $(1, 0)$.

When counted with multiplicity there are two points of Ω. The associated cohomology product is the 4th power of the hyperplane class in $\mathrm{Gr}(2, 4)$ which is 2 points. It follows from Theorem 4.2 that Ω is set-theoretically exactly one point V counted with multiplicity 2.

It is easy to see that V admits a first order deformation in Ω. The deformation is given by the $\mathbb{C}$-span of $x + \varepsilon$ and $x^3 + 2 - 3\varepsilon\, x^2$ over $\mathbb{C}[\varepsilon]/(\varepsilon^2)$. Indeed we have

$$\mathrm{Wr}\left(x + \varepsilon, -1 - \frac{x^3}{2} + \frac{3\varepsilon\, x^2}{2}\right) = x^3 - 1 \mod \varepsilon^2.$$

5 Critical points of master functions

Let $l_1, \dots, l_N$ be nonnegative integers and $z_1, \dots, z_n$ distinct complex numbers. For $s = 1, \dots, n$, fix nonnegative integers $m_s(1), \dots, m_s(N)$. Define polynomials $T_1, \dots, T_N$ by the formula $T_i = \prod_{s=1}^n (x - z_s)^{m_s(i)}$. *The master function* Φ

associated to this data is the rational function

$$\Phi(t_j^{(i)}) = \prod_{i=1}^{N}\prod_{j=1}^{l_i} T_i(t_j^{(i)})^{-1} \prod_{i=1}^{N-1}\prod_{j=1}^{l_i}\prod_{k=1}^{l_{i+1}}(t_j^{(i)} - t_k^{(i+1)})^{-1} \prod_{i=1}^{N}\prod_{1\le j<k\le l_i}(t_j^{(i)} - t_k^{(i)})^2$$

of $l_1 + \cdots + l_N$ variables $t_1^{(1)}, \dots, t_{l_1}^{(1)}, \dots, t_1^{(N)}, \dots, t_{l_N}^{(N)}$ considered on the set of points where

- the numbers $t_1^{(i)}, \dots, t_{l_i}^{(i)}$ are distinct;
- the sets $\{t_1^{(i)}, \dots, t_{l_i}^{(i)}\}$ and $\{t_1^{(i+1)}, \dots, t_{l_{i+1}}^{(i+1)}\}$ do not intersect;
- the sets $\{t_1^{(i)}, \dots, t_{l_i}^{(i)}\}$ and $\{z_1, \dots, z_n\}$ do not intersect.

Let $l_0 = l_{N+1} = 0$. Define $c_i = i - 1 + l_i - l_{i-1} + \sum_{j=1}^{i-1}\sum_{s=1}^{n} m_s(j)$ for $i = 1, \dots, N+1$.

Lemma 5.1. *If either $c_i < 0$ for some $i = 1, \dots, N+1$ or $c_i = c_j$ for some $1 \le i < j \le N+1$, then Φ has no critical points.*

Proof. Let $(t_j^{(i)})$ be a critical point of Φ. Set $y_i = \prod_{j=1}^{l_i}(x - t_j^{(i)})$ for $i = 1, \dots, N$. Then y_i divides $\mathrm{Wr}(y_i', T_i\, y_{i-1}\, y_{i+1})$, where as usual we set $y_{N+1} = y_0 = 1$.

We have $c_i = i - 1 + \deg y_i - \deg y_{i-1} + \sum_{j=1}^{i-1}\deg T_j$. By part (3) of Theorem 3.5, $c_1, \dots, c_{N+1}$ are pairwise distinct non-negative numbers. □

Assume $c_1, \dots, c_{N+1}$ are pairwise distinct nonnegative integers. Let $u \in \Sigma_{N+1}$ be the permutation such that $c_{u(1)} > c_{u(2)} > \cdots > c_{u(N+1)}$. Let $d = \max\{c_1, \dots, c_{N+1}\}$.

For $z \in \{z_1, \dots, z_n\}$ define the ramification sequence $\boldsymbol{a}(z)$ by the rule

$$a_j(z) = \sum_{\ell=j}^{N} m_{N+1-\ell}(z), \quad j = 1, \dots, N+1.$$

Define the ramification sequence $\boldsymbol{a}(\infty)$ at ∞ by the rule

$$a_j(\infty) = d - (N+1) + j - c_{u(j)}, \quad j = 1, \dots, N+1.$$

Proposition 5.2. *The master function Φ of this section is the same as the master function of the basic situation of Section 4 associated with the space $W = \mathbb{C}_d[x]$, ramification sequences $\boldsymbol{a}(z_1), \dots, \boldsymbol{a}(z_n), \boldsymbol{a}(\infty)$, and the permutation u^{-1} in Σ_{N+1}.*

□

6 Appendix

6.1 Equation $\mathrm{Wr}(y, \tilde{y}) = T$

Lemma 6.1. *Let A be an algebra over $\mathbb{C}$. Let $y, T \in A[x]$.*

(1) *If the equation $\mathrm{Wr}(y, \tilde{y}) = T$ has a solution $\tilde{y} \in A[x]$ then y divides $\mathrm{Wr}(y', T)$.*

(2) *If the ideal (y, y') is equal to $A[x]$ and y divides $\mathrm{Wr}(y', T)$, then the equation $\mathrm{Wr}(y, \tilde{y}) = T$ has a solution $\tilde{y} \in A[x]$.*

Proof. If $T = \mathrm{Wr}(y, \tilde{y})$, then $\mathrm{Wr}(y', \mathrm{Wr}(y, \tilde{y})) = y''(y'\tilde{y} - y\tilde{y}') - y'(y''\tilde{y} - y\tilde{y}'') = y(y'\tilde{y}'' - y''\tilde{y}')$, and y divides $\mathrm{Wr}(y', T)$.

For the other direction, write $T = ay + by'$ for suitable polynomials a, b. Then $T = (a + b')\, y + \mathrm{Wr}(y, b)$. We have $\mathrm{Wr}(y', T) = \mathrm{Wr}(y', y(a + b')) + \mathrm{Wr}(y', \mathrm{Wr}(y, b))$. The last term is divisible by y. If y divides $\mathrm{Wr}(y', T)$, then y divides $\mathrm{Wr}(y',\ y\,(a+b'))$. This implies that y divides $(y')^2(a+b')$. Writing $c\, y' = 1 - dy$ for suitable polynomials c and d, we deduce that y divides $(a + b')$. Thus, $y(a + b') = y^2 e$ for a suitable polynomial e. Let f be a polynomial whose derivative is $-e$. Then $\mathrm{Wr}(1, f) = e$, and $\mathrm{Wr}(y, yf) = y(a + b')$. Finally, $\mathrm{Wr}(y, yf + b) = y(a + b') + \mathrm{Wr}(y, b) = T$. □

Lemma 6.2. *Suppose $y, f \in A[x]$ and y is monic.*

(1) *There exist unique $q, r \in A[x]$ such that $f = y\, q + r$ and $\deg r < \deg y$.*

(2) *Suppose that $y = \prod_{j=1}^{m} (x - t_j)$, $t_j \in A$ and $t_j - t_k$ are units in A for $j \neq k$. Then the ideal in A generated by coefficients of the polynomial r in (1), coincides with the ideal in A generated by $f(t_1), \dots, f(t_m)$.*

Proof. Existence and uniqueness of q and r follow from the monicity of y and long division.

To prove the second part, write $f(t_j) = r_{m-1} t_j^{m-1} + \cdots + r_0$. We deduce from Cramer's rule and the formula for the Vandermonde determinant that the coefficients of r lie in the ideal generated by $f(t_1), \dots, f(t_m)$. □

Lemma 6.3. *Let $(A, \mathfrak{m})$ be a local ring and $y_1, y_2 \in A[x]$ with y_1 monic. Suppose that the ideal generated by the reductions $(\bar{y}_1, \bar{y}_2) = (A/\mathfrak{m})[x]$. Then the ideal (y_1, y_2) equals $A[x]$.*

Proof. The A-module $A[x]/(y_1)$ is a finite A-module because y_1 is monic. The quotient $M = A[x]/(y_1, y_2)$ of $A[x]$ by the ideal (y_1, y_2) is therefore a finite A-module. Now, $M \otimes A/\mathfrak{m} = (A/\mathfrak{m})[x] / (\bar{y}_1, \bar{y}_2) = 0$ by hypothesis. By Nakayama's Lemma ([M], Theorem 4.8), we conclude that $M = 0$. □

Lemma 6.4. *Let A be an algebra, $y = \prod_{j=1}^{m} (x - t_j) \in A[x]$ with $t_j \in A$ and $T = \prod_{l=1}^{n} (x - z_l) \in A[x]$ with $z_l \in A$. Then*

(1) *The ideal* $(y,\ y')$ *is equal to* $A[x]$ *if and only if* $t_i - t_j$ *are units in* A *for all* $i < j$.

(2) *The ideal* $(T,\ y)$ *is equal to* $A[x]$ *if and only if* $t_j - z_l$ *are units in* A *for all* j, l.

(3) *Assume that the ideals* $(y,\ y') = A[x]$ *and* $(T,\ y) = A[x]$. *Then,* y *divides* $\mathrm{Wr}(y', T)$ *if and only if the following system of equations holds:*

$$\sum_{l\in\{1,\dots,m\}-j} \frac{2}{t_j - t_l} - \sum_{l=1}^{n} \frac{1}{t_j - z_l} = 0, \quad j = 1, \dots, m. \tag{6.1}$$

Notice that the system of equations (6.1) coincides with the critical point equations for the function

$$\Phi(t_1, \dots, t_m) = \prod_{1\le i<j\le m} (t_i - t_j)^2 \prod_{j=1}^{m}\prod_{l=1}^{n}(t_j - z_l)^{-1} = \prod_{1\le i<j\le m} (t_i - t_j)^2 \prod_{j=1}^{n} T(t_j)^{-1}$$

of variables $t_1, \dots, t_m$.

Proof. If the ideal $(y, y') = A[x]$, write $1 = ay + by'$ and substitute $x = t_i$ to conclude that $t_i - t_j$ is invertible for $i \neq j$.

For the other direction, let $M = A[x]/(y, y')$. Clearly,

$$M/\mathfrak{m}M = (A/\mathfrak{m})[x]/(\bar{y}, \bar{y}') = 0,$$

which is guaranteed by the assumption and the standard theory of fields. Since y is monic, by Lemma 6.3, $M = 0$. The assertion (2) is proved similarly.

It is easy to see that (3) follows from Lemma 6.2 and Lemma 6.1. □

Lemma 6.5. *For* $y_1, y_2, T \in A[x]$, *assume that the ideal* $(y_1,\ y_2) = A[x]$ *and* y_1 *divides* $y_2 T$. *Then* y_1 *divides* T.

Proof. Write $1 = ay_1 + by_2$. Hence $T = aTy_1 + bTy_2$. The last two terms are divisible by y_1. Hence y_1 divides T. □

6.2 Proof of Lemma 3.2

The implication (1) ⇒ (3) is immediate.

If (2) holds, then V is a finitely generated module which is a direct summand of a free module. This implies that it is free (being a direct summand implies that it is projective and projective modules over local rings are free). The morphism $V \otimes A/\mathfrak{m} \to \mathbb{C}[x]$ is a direct summand of the isomorphism $A[x] \otimes A/\mathfrak{m} \to \mathbb{C}[x]$ and hence is injective. This gives (1).

We prove (1) and (2) assuming (3). Pick a large integer d so that $V \subset A_d[x]$. By assumption we can then find $v_1, \dots, v_{d+1-k} \in A_d[x]$ such that the collection $\bar{u}_1, \dots, \bar{u}_k, \bar{v}_1, \dots, \bar{v}_{d+1-k}$ form a basis for the $\mathbb{C}$-vector space $\mathbb{C}_d[x]$. This implies that the determinant of the change of basis matrix from the standard basis of

$A_d[x]$ to $u_1, \ldots, u_k, v_1, \ldots, v_{d+1-k}$ does not vanish upon reduction to the residue field. Therefore, the determinant is a unit in A. Hence $u_1, \ldots, u_k, v_1, \ldots, v_{d+1-k}$ form a free basis in $A_d[x]$. This proves (1) and (2), where for (2) we let $M = \mathrm{Span}_A(v_1, \ldots, v_{d+1-k}) \oplus x^{d+1}A[x] \subset A[x]$.

6.3 Multiplicity

We will recall the algebro-geometric definitions of multiplicity from [F], Section 1.5. In this section we will need to consider local rings whose residue field may be different from $\mathbb{C}$.

Let X be an irreducible algebraic scheme. The geometric multiplicity of X, denoted by $m(X)$, is the length of the local ring of X at its generic point. Explicitly, if $\mathrm{Spec}(A)$ is an affine open subset of X, then A has exactly one minimal prime ideal. Denote it by $\mathfrak{p}$. The localisation $A_\mathfrak{p}$ is an Artin ring and is therefore of finite length. The integer $m(X)$ is the length of $A_\mathfrak{p}$. A more practical way of computing $m(X)$ is obtained from Proposition 6.7.

Example. Consider the geometric multiplicity of the so-called "doubled line" $\mathrm{Spec}(\mathbb{C}[x, y]/(x^2))$. This has exactly one minimal prime ideal, namely (x). The localisation at this minimal prime ideal is the ring $\mathbb{C}(y)[x]/(x^2)$, which is of length 2.

Now, we will discuss properties of geometric multiplicity, linking it to "multiplicity in the transversal direction". The first property is (see [F], Example A.1.1):

Proposition 6.6. *If X is an irreducible 0-dimensional scheme (i.e., a fat point), then $m(X)$ is the dimension over $\mathbb{C}$ of the ring of functions of X,*

$$m(X) = \dim_\mathbb{C} \Gamma(X, \mathcal{O}_X).$$

In general, if X is an irreducible scheme, then X is reduced if and only if its geometric multiplicity is 1.

The following proposition follows from Lemma 1.7.2 in [F].

Proposition 6.7. *Suppose that X is an irreducible subscheme of $\mathbb{C}^n$. Let X_{red} be the reduced subscheme corresponding to X. The subscheme X_{red} can be considered to be a closed subscheme of X. Let U be the smooth locus of X_{red}. Let H be a hyperplane in $\mathbb{C}^n$ which meets U transversally at a point $x \in U$. Let D be the irreducible component of $X \cap H$ which contains x (there is exactly one such irreducible component). Then,*

$$m(X) = m(D).$$

Iterating this procedure, we obtain the following statement. Suppose T is a plane in $\mathbb{C}^n$ of dimension complementary to $\dim X$*, which meets U transversally at a point $x \in U$ (there could be other points of intersection). Then, the multiplicity of X is equal to the dimension over $\mathbb{C}$ of the localization at x of the algebra of functions on the scheme $X \cap H$.*

There is one other standard property of multiplicity that we will need. Recall that a smooth morphism between schemes is a flat morphism with smooth fibers.

Proposition 6.8. *Let $f: X \to Y$ be a smooth morphism between irreducible schemes. Then $m(X) = m(Y)$.*

Proof. We will use the notations and definitions of [F]. Let X_{red} and Y_{red} be the reduced schemes underlying X and Y. Then by definition $[X] = m(X)[X_{\text{red}}]$ and $[Y] = m(Y)[Y_{\text{red}}]$. The smoothness of f tells us that $f^{-1}(Y_{\text{red}})$ is reduced and hence $f^{-1}(Y_{\text{red}}) = X_{\text{red}}$. Clearly $f^{-1}(Y) = X$. Apply Lemma 1.7.1 in [F] to see that $f^*[Y] = [f^{-1}(Y)]$. This gives $m(Y)[X_{\text{red}}] = m(X)[X_{\text{red}}]$ and therefore $m(X) = m(Y)$. □

6.4 Multiplicity in intersection theory

Irreducible subvarieties $X_1, \ldots, X_r$ of a smooth variety X are said to intersect properly, provided each irreducible component of $X_1 \cap \cdots \cap X_r$ is of dimension $\dim X - \sum_{j=1}^r(\dim X - \dim X_j)$. We will use the following basic result.

Denote the smooth locus of X_i by X_i^o. Suppose that $X_1, \ldots, X_r$ intersect properly in a finite set, that is, the expected dimension of the intersection is 0. Suppose that

$$X_1 \cap \cdots \cap X_r = X_1^o \cap \cdots \cap X_r^o.$$

Then we have an equality of cohomology classes in $H^*(X)$,

$$\prod_{i=1}^r [X_i] = c\,[\text{pt}],$$

where c is the sum of the multiplicities of the irreducible components of the scheme theoretic intersection $X_1 \cap \cdots \cap X_r$ and [pt] the class of a point.

This statement follows from [F], Proposition 7.1.

6.5 Standard results in the theory of schemes

For a scheme X and a $\mathbb{C}$-algebra A, we let

$$X(A) = \text{Hom}(\text{Spec}(A), X).$$

If A is a local ring, and $s \in X(A)$, then we denote the induced point in $X(\mathbb{C}) = X(A/\mathfrak{m})$ by $\bar{s}$. If $x \in X(A/\mathfrak{m})$ is given, we let $X_x(A) = \{s \in X(A) \mid \bar{s} = x\}$. For $s \in X_x(A)$ there corresponds a local homomorphism of local rings $\mathcal{O}_{X,x} \to A$, where $\mathcal{O}_{X,x}$ is the local ring of X at x.

Lemma 6.9. *Let $f: X \to Y$ be a finite morphism of schemes. Then, f is a closed immersion if and only if for every local ring A, the induced mapping $X(A) \to Y(A)$ is injective.*

Proof. If $f: X \to Y$ makes X a subscheme of Y, then clearly $X(A) \to Y(A)$ is injective.

To go the other way, let $x \in X$ and $y = f(x)$. By taking $A = \mathbb{C}$, we see that $f^{-1}(y)$ is the singleton $\{x\}$. Denote the local ring of X at x by $(\mathcal{O}_{X,x}, m_x)$ and that of Y at y by $(\mathcal{O}_{Y,y}, m_y)$.

Now let $A = \mathbb{C}[\varepsilon]/(\varepsilon^2)$. Consider the induced mapping $X_x(A) \to Y_y(A)$. This is once again injective by hypothesis. It is a basic fact that $X_x(A) = \mathrm{Hom}(m_x/m_x^2, \mathbb{C})$ and $Y_y(A) = \mathrm{Hom}(m_y/m_y^2, \mathbb{C})$. So the hypothesis implies that $\mathrm{Hom}(m_x/m_x^2, \mathbb{C}) \to \mathrm{Hom}(m_y/m_y^2, \mathbb{C})$ is injective or that the natural morphism $m_y/m_y^2 \to m_x/m_x^2$ is surjective. By [H], II.7.4, we conclude that the map $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is surjective. This shows that $X \to Y$ is a closed inclusion. □

Lemma 6.10. *Let $f: X \to Y$ be a closed immersion of schemes. Then f is an isomorphism if and only if $f: X(A) \to Y(A)$ is surjective for every local ring A.*

Proof. If f is an isomorphism, then $X(A) \to Y(A)$ is clearly bijective for any local ring A.

To go the other way, let $x \in X$ and $y = f(x)$. Denote the local ring of X at x by $(\mathcal{O}_{X,x}, m_x)$ and that of Y at y by $(\mathcal{O}_{Y,y}, m_y)$.

Suppose that y is in an open affine subset $\mathrm{Spec}(B) \subseteq Y$. Let $\mathfrak{m}$ be the maximal ideal of $\mathrm{Spec}(B)$ corresponding to the point y. Then $\mathcal{O}_{Y,y}$ is the same as the localization B_m of B at m. The natural map $B \to B_m$ gives a point η in $\mathrm{Spec}(B)(\mathcal{O}_{Y,y}) \subseteq Y(\mathcal{O}_{Y,y})$. Clearly $\overline{\eta} = y$ and therefore $\eta \in Y_y(\mathcal{O}_{Y,y})$. The map of local rings corresponding to η is the identity map $\mathcal{O}_{Y,y} \to \mathcal{O}_{Y,y}$

By the given hypothesis, there exists $\theta \in X(\mathcal{O}_{Y,y})$ such that $f(\theta) = \eta$. The reduction of this point is $x \in X(\mathbb{C})$ because the reduction has to sit over $y \in Y(\mathbb{C})$.

Therefore we obtain a diagram

$$\begin{array}{ccc} \mathcal{O}_{Y,y} & \xrightarrow{f} & \mathcal{O}_{X,x} \\ & \underset{=}{\searrow} & \downarrow \theta \\ & & \mathcal{O}_{Y,y} \end{array}$$

Hence $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is both injective and surjective whence an isomorphism. □

The following proposition is standard and follows from the description of Schubert varieties as degeneracy loci (cf. [F], chapter 14).

Proposition 6.11. *Let A be a local ring and W a $\mathbb{C}$-vector space of rank $d+1$. Then,* $\mathrm{Gr}(N+1, W)(A)$ *is the set of free submodules $V \subset W \otimes A$ of rank $N+1$, such that $V \otimes A/\mathfrak{m} \to W$ is injective.*

The subset $\Omega_{\boldsymbol{a}}^o(\mathcal{F})(A) \subseteq \mathrm{Gr}(N+1, W)(A)$ consists of submodules V such that there exists an A-basis $u_1, \ldots, u_{d+1}$ of $W \otimes A$ with the following properties:

- *F_i is the A-span of $u_1, \ldots, u_i$ for $i = 1, \ldots, d+1$;*
- *V is the A-span of the elements $u_{d-N+j-a_j}$ for $j = 1, \ldots, N+1$.*

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Some explicit solutions to the Riemann–Hilbert problem

Philip Boalch

École Normale Supérieure
45 rue d'Ulm, 75005 Paris, France
email: `boalch@dma.ens.fr`

Dedicated to Andrey Bolibruch

Abstract. Explicit solutions to the Riemann–Hilbert problem will be found realising some irreducible non-rigid local systems. The relation to isomonodromy and the sixth Painlevé equation will be described.

Contents

1 Introduction

Unfortunately, to say that a particular Riemann–Hilbert problem is "solvable", one usually means that *there exists a solution* rather than that one is actually able to solve the problem explicitly.

In this article we will confront the problem of explicitly solving the Riemann–Hilbert problem directly, for irreducible representations (so we already know the problem is "solvable"). We will describe how one soon becomes embroiled in isomonodromic deformation equations, from which it is easy to see the difficulty: in the simplest non-trivial case the isomonodromy equations reduce to the sixth Painlevé equation P_{VI} and one knows that generic solutions of P_{VI} cannot be written explicitly in terms of classical special functions.

However there are some explicit solutions to P_{VI} and our aim will be to write down some new solutions controlling isomonodromic deformations of non-rigid rank two Fuchsian systems on the four-punctured sphere. The cases we will study here will have monodromy group equal to either the binary tetrahedral or octahedral group (the icosahedral case having been studied in [5]), or to one of the triangle groups Δ_{237} or Δ_{238}.

Previously a tetrahedral and an octahedral solution of P_{VI} have been constructed by Hitchin [13] and (up to equivalence) independently by Dubrovin [9]. Moreover with hindsight we see there are three other such solutions in the work of Andreev and Kitaev [1], [18]. Here we will classify all such solutions and find an explicit solution in each of the new cases that appear.

Amongst the solutions which look to be new (i.e. to the best of the author's knowledge have not previously appeared) there are five octahedral solutions including one of genus one, and two 18 branch genus one solutions with monodromy group Δ_{237}. The largest octahedral solution has sixteen branches which is (currently) the largest known genus zero solution (those with more branches in [5] having higher genus) and we will show it is equivalent to a solution with monodromy group Δ_{238}.

The results of Sections 3 and 4 will be of particular interest to people interested in constructing linear differential equations with algebraic solutions (cf. e.g. [17], [3], [27], [4]). Indeed Tables 1 and 3 may be interpreted as the analogue for rank two Fuchsian systems with four poles on $\mathbb{P}^1$, of the tetrahedral and octahedral parts of Schwarz's famous list [25] of hypergeometric equations with algebraic bases of solutions.

2 From Riemann–Hilbert to Painlevé

Consider a logarithmic connection ∇ on the trivial rank n complex vector bundle over the Riemann sphere with singularities at points $a_1, \ldots, a_m$. Choosing a coordinate z on the sphere (in which $a_m = \infty$ say), this amounts to giving the Fuchsian system of differential equations $\nabla_{d/dz}$ which will have the following form:

$$\frac{d}{dz} - A(z); \qquad A(z) = \sum_{i=1}^{m-1} \frac{A_i}{z - a_i} \tag{1}$$

for complex $n \times n$ matrices A_i. The original Riemann–Hilbert map is the map which takes such a Fuchsian system to its monodromy data: restricting ∇ to the punctured sphere

$$\mathbb{P}^* := \mathbb{P}^1 \setminus \{a_1, \ldots, a_m\}$$

yields a nonsingular holomorphic connection and taking its monodromy yields a representation

$$\rho \in \mathrm{Hom}(\pi_1(\mathbb{P}^*), G)$$

where $G = \mathrm{GL}_n(\mathbb{C})$. The Riemann–Hilbert problem is the following: given $a_1, \dots a_m$ and ρ can we find such a connection ∇ with monodromy equal to ρ?

Upon choosing simple loops γ_i in $\mathbb{P}^*$ around a_i generating $\pi_1(\mathbb{P}^*)$ and such that $\gamma_m \circ \cdots \circ \gamma_1$ is contractible one sees that for each m-tuple of points $\boldsymbol{a} = (a_1, \dots, a_m)$ the Riemann–Hilbert map amounts to a map between the following spaces:

$$\left\{(A_1, \dots, A_m) \mid \sum A_i = 0\right\} \xrightarrow{\mathrm{RH}_{\boldsymbol{a}}} \left\{(M_1, \dots, M_m) \mid M_m \dots M_1 = 1\right\} \qquad (2)$$

where $M_i = \rho(\gamma_i) \in G$. The Riemann–Hilbert problem then becomes: given a point $\boldsymbol{M} = (M_1, \dots, M_m)$ on the RHS of (2), are there matrices $\boldsymbol{A} = (A_1, \dots A_m)$ with $\sum A_i = 0$ on the LHS such that $\mathrm{RH}_{\boldsymbol{a}}(\boldsymbol{A}) = \boldsymbol{M}$?

Remark 1. So far we have ignored the questions of choosing a basepoint for $\pi_1(\mathbb{P}^*)$ and the choice of basis of the fibre at the basepoint. However it is immediate that if we have a solution $\mathrm{RH}_{\boldsymbol{a}}(\boldsymbol{A}) = \boldsymbol{M}$ (defined with respect to some choice of basepoint/basis) then conjugating the matrices A_i by some constant matrix $g \in G$ corresponds to conjugating the monodromy matrices M_i as well. Thus the Riemann–Hilbert problem is independent of the choice of basepoint/basis since these just move to conjugate representations.

Some fundamental work on the Riemann–Hilbert problem was done by Schlesinger [24]. He considered the question of constructing new Riemann–Hilbert solutions from a given solution $\mathrm{RH}_{\boldsymbol{a}}(\boldsymbol{A}) = \boldsymbol{M}$, in two ways:

1) Schlesinger examined the fibres of the Riemann–Hilbert map and defined "Schlesinger transformations", which move $\boldsymbol{A}$ within the fibres (cf. also [16]). Roughly speaking generic fibres are discrete and correspond to certain integer shifts in the eigenvalues of the matrices A_i; geometrically these Schlesinger transformations amount to rational gauge transformations with singularities at the poles of the Fuchsian system.

2) Schlesinger also found how the matrices $\boldsymbol{A}$ can be varied as one moves the pole positions $\boldsymbol{a}$ in order to realise the same monodromy data $\boldsymbol{M}$. (Locally – for small deformations of $\boldsymbol{a}$ – this makes sense as one can use the same loops generating $\pi_1(\mathbb{P}^*)$; globally one should drag the loops around with the points $\boldsymbol{a}$, so on returning $\boldsymbol{a}$ to their initial configuration ρ may have changed by the action of the mapping class group of the m-pointed sphere.) He discovered that if the matrices A_i satisfy the following nonlinear differential equations, now known as the Schlesinger equations, then locally the monodromy data is preserved (up to overall conjugation):

$$\frac{\partial A_i}{\partial a_j} = \frac{[A_i, A_j]}{a_i - a_j} \quad \text{if } i \neq j, \text{ and } \quad \frac{\partial A_i}{\partial a_i} = -\sum_{j \neq i} \frac{[A_i, A_j]}{a_i - a_j}. \qquad (3)$$

In the generic case such an "isomonodromic deformation" necessarily satisfies these equations (up to conjugation). This gives a hint at the difficulty of the Riemann–

Hilbert problem: even if one knows a solution for some configuration of pole positions, one must integrate some nonlinear differential equations to obtain solutions for a deformed configuration.

This also gives a hint at how one might find some interesting solutions to the Riemann–Hilbert problem. Namely since one can move the pole positions one may consider degenerations into systems with fewer poles (for which the problem should be easier). Using solutions to these degenerate Riemann–Hilbert problems one can get asymptotics for the original solution to the Schlesinger equations and in good circumstances this enables computation of the solution. This is in effect what we will do below (using the analysis of the degenerations in [23], part II, and [15]).

Suppose we fix an irreducible representation $\rho \in \mathrm{Hom}(\pi_1(\mathbb{P}^*), G)$. Let $\mathcal{C}_i \subset G$ be the conjugacy class containing $M_i = \rho(\gamma_i)$ which we will suppose for simplicity is regular semisimple, although this is not strictly necessary. (We are thus considering "generic" representations.)

Since ρ is irreducible we know [2] there exists some Riemann–Hilbert solution $\mathrm{RH}_{\boldsymbol{a}}(\boldsymbol{A}) = \boldsymbol{M}$. Let $\mathcal{O}_i \subset \mathfrak{g}$ be the adjoint orbit of A_i (in the Lie algebra of $n \times n$ complex matrices). By genericity we know $\exp(2\pi\sqrt{-1}\mathcal{O}_i) = \mathcal{C}_i$. Indeed if in the Riemann–Hilbert map we restrict to $A_i \in \mathcal{O}_i$ then one has $M_i \in \mathcal{C}_i$. Also, as mentioned above, the map is equivariant under diagonal conjugation and so there is a "reduced Riemann–Hilbert map":

$$\boldsymbol{\mathcal{O}} := \mathcal{O}_1 \times \cdots \times \mathcal{O}_m /\!\!/ G \xrightarrow{\nu_{\boldsymbol{a}}} \mathcal{C}_1 \circledast \cdots \circledast \mathcal{C}_m /\!\!/ G =: \boldsymbol{\mathcal{C}} \tag{4}$$

where the space $\boldsymbol{\mathcal{O}}$ is the quotient of $\{(A_1, \ldots, A_m) \mid A_i \in \mathcal{O}_i, \sum A_i = 0\}$ by overall conjugation by G and $\boldsymbol{\mathcal{C}}$ is the quotient of $\{(M_1, \ldots, M_m) \mid M_i \in \mathcal{C}_i, M_m \ldots M_1 = 1\}$ by overall conjugation by G. Generally this map $\nu_{\boldsymbol{a}}$ is an injective holomorphic symplectic map between complex symplectic manifolds of the same dimension.

The simplest case is when the representation is rigid, i.e. when the expected dimensions of both sides of (4) is zero. Then one knows the RHS of (4) consists of precisely one point and the LHS (at most) one point.

Our basic strategy is to look at the next simplest case, with the aim of degenerating into the rigid case. Since the spaces are symplectic, this corresponds to complex dimension two, i.e. both sides of (4) are complex surfaces.

The principal example of such "minimally non-rigid" systems occurs if we look at rank two systems with four poles on the sphere (i.e. $n = 2, m = 4$). Without loss of generality (by tensoring by logarithmic connections on line-bundles) one can work with $G = \mathrm{SL}_2(\mathbb{C})$ rather than $\mathrm{GL}_2(\mathbb{C})$ and, using automorphisms of the sphere we can fix three of the poles at $0, 1, \infty$ and label the remaining pole position t. Thus we are considering systems of the form

$$\frac{d}{dz} - \left(\frac{A_1}{z} + \frac{A_2}{z-t} + \frac{A_3}{z-1}\right), \quad A_i \in \mathfrak{g} := \mathfrak{sl}_2(\mathbb{C}). \tag{5}$$

By convention we denote the eigenvalues of A_i by $\pm\theta_i/2$ for $i = 1, 2, 3, 4$. Schlesinger's equations imply that the residue $A_4 = -\sum_1^3 A_i$ at infinity remains fixed; we will conjugate the system so that $A_4 = \frac{1}{2}\,\mathrm{diag}(\theta_4, -\theta_4)$. The remaining conjugation freedom is then just conjugation by the one-dimensional torus $T := \mathrm{diag}(a, 1/a)$, $a \in \mathbb{C}^*$; the space of such systems is then three dimensional (quotienting by T yields the surface $\boldsymbol{\mathcal{O}}$).

Following [16] (pp. 443–446) one may choose certain coordinates x, y, k on this space of systems and write down what Schlesinger's equations become. One obtains a pair of coupled first-order nonlinear differential equations in x, y (not dependent on k) and an equation for k of the form $\frac{dk}{dt} = f(y,t)k$. The coordinate k corresponds to the torus action, which we can forget about since we are happy to consider Fuchsian systems up to conjugation. Eliminating x from the coupled system yields the sixth Painlevé equation

$$\frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt} + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{(t-1)}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right),$$

where the constants $\alpha, \beta, \gamma, \delta$ are related to the θ-parameters as follows:

$$\alpha = (\theta_4 - 1)^2/2, \quad \beta = -\theta_1^2/2, \quad \gamma = \theta_3^2/2, \quad \delta = (1 - \theta_2^2)/2. \tag{6}$$

Since we will want to go back from a solution of $\mathrm{P_{VI}}$ to an explicit isomonodromic family of Fuchsian systems, we will give the explicit formulae for the matrix entries of the system in terms of y and y' in Appendix A.

Now the bad news is that most solutions to $\mathrm{P_{VI}}$ cannot be written in terms of classical special functions. From Watanabe's work [28] one knows that either a solution is non-classical or it is a Riccati solution (corresponding to a reducible or rigid monodromy representation ρ) or the solution $y(t)$ is an algebraic function.

Since we are interested in explicit solutions corresponding to irreducible non-rigid representations, the only possibility is to seek algebraic solutions to $\mathrm{P_{VI}}$, in other words solutions defined implicitly by equations of the form

$$F(y,t) = 0$$

for polynomials F in two variables. We can rephrase this more geometrically:

Definition 2. An algebraic solution of $\mathrm{P_{VI}}$ consists of a triple (Π, y, t) where Π is a compact (possibly singular) algebraic curve and y, t are rational functions on Π such that:

• $t\colon \Pi \to \mathbb{P}^1$ is a Belyi map (i.e. t expresses Π as a branched cover of $\mathbb{P}^1$ which only ramifies over $0, 1, \infty$), and

• using t as a local coordinate on Π away from ramification points, $y(t)$ should solve $\mathrm{P_{VI}}$, for some value of the parameters $\alpha, \beta, \gamma, \delta$.

Indeed given an algebraic solution in the form $F(y,t) = 0$ one may take Π to be the closure in $\mathbb{P}^2$ of the affine plane curve defined by F. That t is a Belyi map on Π

follows from the Painlevé property of P_{VI}: solutions will only branch at $t = 0, 1, \infty$ and all other singularities are just poles. The reason we prefer this reformulation is that often the polynomial F is quite complicated and parameterisations of the plane curve defined by F are usually simpler to write down. (The polynomial F can be recovered as the minimal polynomial of y over $\mathbb{C}(t)$, since $\mathbb{C}(y, t)$ is a finite extension of $\mathbb{C}(t)$.)

We will say the solution curve Π is 'minimal' or an 'efficient parameterisation' if y generates the field of rational functions on Π, over $\mathbb{C}(t)$, so that y and t are not pulled back from another curve covered by Π (i.e. that Π is birational to the curve defined by F).

The main invariants of an algebraic solution are the genus of the (minimal) Painlevé curve Π and the degree of the corresponding Belyi map t (the number of branches the solution has over the t-line).

Now the basic question is: what representations ρ can we start with in order to obtain an algebraic solution to P_{VI}? Well, the solution must have only a finite number of branches and so we can start by looking for finite branching solutions, and hope to prove in each case that the solution is actually algebraic.

The important point is that one can read off the branching of the solution y as t moves around loops in the three-punctured sphere $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ in terms of the corresponding linear representations ρ. One finds (cf. e.g. [5], section 4) that ρ transforms according to the natural action of the pure mapping class group (which is isomorphic to $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\})$ and thus to the free group on two-letters $\mathcal{F}_2$). Explicitly the generators w_1, w_2 of $\mathcal{F}_2$ act on the monodromy matrices $\boldsymbol{M}$ via $w_i = \omega_i^2$ where ω_i fixes M_j for $j \neq i, i+1$, $(1 \leqslant j \leqslant 4)$ and

$$\omega_i(M_i,\ M_{i+1}) = (M_{i+1},\ M_{i+1} M_i M_{i+1}^{-1}). \tag{7}$$

(Incidentally the geometric origins of this in the context of P_{VI} can be traced back at least to Malgrange's work [22] on the global properties of the Schlesinger equations.) The full classification of the representations ρ living in finite orbits of this action is still open, but there are some obvious ones: namely if ρ takes values in a finite subgroup of $SL_2(\mathbb{C})$ then the $\mathcal{F}_2$ orbit will clearly be finite.

Thus the program is to take such a finite subgroup $\Gamma \subset G$, and go through the possible representations $\rho\colon \pi_1(\mathbb{P}^*) \to \Gamma$ (whose image generates Γ say) and find the corresponding P_{VI} solutions. There are two main problems to overcome in completing this program:

1) There are lots of such representations (even up to conjugation), for example for the binary tetrahedral group from [11] one knows there are 520 conjugacy classes of triples of generators.

2) We still need to find the P_{VI} solution explicitly.

For 1) we proceed as in [5]: by using Okamoto's affine F_4 symmetry group of P_{VI} we can drastically reduce the number of classes that arise for each group. It is worth emphasising that upon applying an Okamoto transformation the monodromy group may well become infinite, and currently there are very few examples of algebraic solutions to P_{VI} which are not equivalent to (or simple deformations of) a solution with finite linear monodromy group (see the final remark of Section 5 below).

For 2) we use Jimbo's asymptotic formula (see [15] and the corrected version in [7], Theorem 4). By looking at the degeneration of the Fuchsian system into systems with only three poles (hypergeometric systems) and using explicit solutions of their Riemann–Hilbert problems, Jimbo found explicit formulae for the leading term in the asymptotic expansion of P_{VI} solutions at zero. Using the P_{VI} equation these leading terms determine the Puiseux expansions of each branch of the solution at zero and, taking sufficiently many terms, these enable us to find the solution completely if it is algebraic.

Philosophically the author views this work as an illustration of the utility of Jimbo's asymptotic formula. An alternative method of constructing solutions of Painlevé VI has been proposed by Kitaev (and Andreev) [20], [1], [18] who call it the "RS" method (see also Doran [8] for similar ideas, as well as Section 5 below and also [21] for closely related ideas of F. Klein). Kitaev [20] conjectures that all algebraic solutions arise in this way and, with Andreev, has found some solutions essentially by starting to enumerate all suitable rational maps along which a hypergeometric system may be pulled back.

One of our original aims was to try to ascertain what algebraic P_{VI} solutions are known, up to equivalence under Okamoto transformations and simple deformation (cf. e.g. [5], Remark 15). In other words the aim was to see how much is known of what might be called the "non-abelian Schwarz list", viewing P_{VI} as the simplest non-abelian Gauss–Manin connection. The result is that, so far, all the algebraic solutions the author has seen have turned out to be related to a finite subgroup of $SL_2(\mathbb{C})$ or to the 237 triangle group (see Section 5 below).[1] As an illustrative example of what can happen consider solution 4.1.7.B of [1]: At first glance we see t is a degree 8 function of the parameter s and so one imagines a solution with 8 branches (and wonders if it is related to one of the eight-branch solutions of [5] or [12] or of Section 4 below). However one easily confirms that in fact

$$y = y_{21} = t + \frac{3\eta_\infty\sqrt{t(t-1)}}{\eta_\infty + 1},$$

so it really only has two branches (it was inefficiently parameterised). In turn one finds (for any value of the constant η_∞) this is equivalent to the well-known solution $y = \sqrt{t}$.

On the other hand Jimbo's formula gives us great control, in that we can often go directly from a linear representation ρ to the corresponding P_{VI} solution. In particular the mapping class group orbit of ρ tells us a priori the number of branches (and lots more) that the solution will have. At some point the author realised (see the introduction to [5]) that there should be more solutions related to the symmetries of the Platonic solids than had already appeared; we have found it to be more efficient to

[1] One might be so bold as to conjecture that there are no others, simply because no others have yet been seen, in spite of the variety of approaches used.

first ascertain directly what solutions arise in this way, than for example to enumerate rational maps. (The author's understanding is that a theorem of Klein implies that the solutions of Sections 3 and 4 below and of [5] will arise via rational pullbacks of a hypergeometric system, but it is not clear if the enumeration started in [1] would ever have found all the corresponding rational maps independently.)

3 The tetrahedral solutions

In this section we will classify the solutions to P_{VI} having linear monodromy group equal to the binary tetrahedral group $\Gamma \subset G = SL_2(\mathbb{C})$. The procedure is similar to that used in [5] for the icosahedral group.

First we examine (as in [5], section 2) the set S of G-conjugacy classes of triples of generators (M_1, M_2, M_3) of Γ (i.e. two triples are identified if they are related by conjugating by an element of G). (Equivalently this is the set of conjugacy classes of representations ρ of the fundamental group of the four-punctured sphere into Γ, once we choose a suitable set of generators.) From Hall's formulae [11] one knows there are 12480 triples of generators of Γ and dividing by 24 (the size of the image in $PSL_2(\mathbb{C})$ of the normaliser of Γ in G) we find that S has cardinality 520. Then we quotient S further by the relation of geometric equivalence (cf. [5], section 4): two representations are identified if they are related by the full mapping class group, or by the set of even sign changes of the four monodromy matrices M_i (with $M_4 = (M_3 M_2 M_1)^{-1}$). One finds there are precisely six such geometric equivalence classes, and by Lemma 9 of [5] this implies there are at most six solutions to P_{VI} with tetrahedral monodromy which are inequivalent under Okamoto's affine F_4 action.

On the other hand we can look at the set of θ-parameters corresponding to the representations in S. Since Okamoto transformations act by the standard $W_a(F_4)$ action on the space of parameters, it is easy to find the set of inequivalent parameters that arise from S, cf. [5], section 3. (Since they are real we can map them all into the closure of a chosen alcove.) We find there are exactly six sets of inequivalent parameters that arise and so there are at least six inequivalent tetrahedral solutions. Combining with the previous paragraph we thus see there are precisely six inequivalent tetrahedral solutions to P_{VI}.

Various data about the six classes and the corresponding P_{VI} solutions are listed in Tables 1 and 2. Table 2 lists a representative set of θ-parameters for each class together with numbers σ_{ij} which uniquely determine a triple M_1, M_2, M_3 in S (and thus the linear representation ρ) for that class with the given θ values, via the formula

$$\mathrm{Tr}(M_i M_j) = 2\cos(\pi\sigma_{ij}).$$

The first two columns of Table 1 list the degree and genus of the P_{VI} solution. The column labelled "Walls" lists the number of affine F_4 reflection hyperplanes the parameters of the solution lie on. The type of the solution enables us to see at a glance

Table 1. Properties of the tetrahedral solutions.

	Degree	Genus	Walls	Type	Alcove Point	n	Group	Partitions
1	1	0	2	ab^2	35, 15, 15, 5	96	1	
2	1	0	3	b^3	30, 10, 10, 10	32	1	
3	2	0	3	b^4-	50, 10, 10, 10	48	S_2	$1, 2$
4	3	0	3	b^4+	40, 0, 0, 0	72	S_3	$3, 2$
5	4	0	2	ab^3	45, 5, 5, 5	128	A_4	3
6	6	0	3	a^2b^2	50, 10, 0, 0	144	A_4	$2^2, 3^2$

Table 2. Representative parameters for the tetrahedral solutions.

	$(\theta_1, \theta_2, \theta_3, \theta_4)$	$(\sigma_{12}, \sigma_{23}, \sigma_{13})$
1	1/2, 0, 1/3, 1/3	1/2, 1/3, 1/3
2	1/3, 0, 1/3, 1/3	1/3, 1/3, 1/3
3	1/3, 2/3, 2/3, 2/3	1/2, 1/3, 1/2
4	2/3, 1/3, 1/3, 2/3	1/2, 1/3, 1/2
5	1/3, 1/3, 1/3, 1/2	1/3, 2/3, 1/3
6	1/2, 1/3, 1/3, 1/2	1/3, 1/2, 1/3

which class a given element of S lies in: Given $M_1, M_2, M_3, M_4 \in \Gamma$ their images in $\mathrm{PSL}_2(\mathbb{C}) \cong \mathrm{SO}_3(\mathbb{C})$ are real rotations and we write an "a" for each rotation by half of a turn, a "b" for each rotation by a third of a turn, and write nothing for each trivial rotation thus obtained. This distinguishes all classes except 3 and 4 which both correspond to four rotations by a third of a turn: each M_i thus has parameter $\theta_i = 1/3$ or $\theta_i = 2/3$. For class 3 there are always an odd number of each type of θ ($1/3$ or $2/3$) so we write a minus, and for class 4 there are always an even number of each type, so we write a plus.

Finally the rest of Table 1 lists the corresponding alcove point (scaled by 60), the number n of elements of S belonging to each class, the monodromy group of the cover $t\colon \Pi \to \mathbb{P}^1$ and the unordered collection of sets of ramification indices of this cover over $t = 0, 1, \infty$ (repeating the last set of indices until three are obtained). Thus for example each solution corresponding to row 6 has indices $(3, 3)$ over two points amongst $\{0, 1, \infty\}$ and indices $(1, 1, 2, 2)$ over the third.

All of the tetrahedral solutions have genus zero so we may take Π to be $\mathbb{P}^1$ with parameter s and write the solutions as functions of s. As in the icosahedral case the solutions with at most 4 branches are closely related to known solutions. For classes 1 and 2 one of the monodromy matrices is projectively trivial and so these rows correspond to pairs of generators of the tetrahedral group, i.e. to the two tetrahedral entries on Schwarz's list of algebraic hypergeometric functions. The corresponding P_{VI} solutions are both just $y = t$ with the parameters as listed in Table 2. As in [5] one finds class 3 contains the solution $y = \pm\sqrt{t}$ (with the parameters as listed in Table 2). Class 4 contains the tetrahedral solution

$$y = \frac{(s-1)(s+2)}{s(s+1)}, \quad t = \frac{(s-1)^2(s+2)}{(s+1)^2(s-2)} \tag{8}$$

on p. 592 of [13] (with the parameters as listed in Table 2) and is equivalent to a solution found independently by Dubrovin [9] (E.31). Also class 5 contains a simple deformation of the four-branch dihedral solution in section 6.1 of [12]:

$$y = \frac{s^2(s+2)}{s^2+s+1}, \quad t = \frac{s^3(s+2)}{2s+1}, \tag{9}$$

that is, this solution is tetrahedral if we use the parameters in Table 2, rather than the parameters (1/2, 1/2, 1/2, 1/2) for which it is dihedral.

Thus we are left with one solution, corresponding to row 6. Using Jimbo's asymptotic formula to compute the Puiseux expansions etc. (as in [7], section 5, especially p. 193) we find the following solution in this class:

Tetrahedral solution 6, 6 branches, $(\theta_1, \theta_2, \theta_3, \theta_4) = (1/2, 1/3, 1/3, 1/2)$:

$$y = -\frac{s\,(s+1)\,(s-3)^2}{3\,(s+3)\,(s-1)^2}, \quad t = -\left(\frac{(s+1)\,(s-3)}{(s-1)\,(s+3)}\right)^3.$$

(We have recently learnt that this is equivalent to solution 4.1.1A in [1].) It is now easy to write down the explicit isomonodromic family of Fuchsian systems in this case, thereby solving the Riemann–Hilbert problem for this class of representations ρ, for an arbitrary configuration of the four pole positions (up to automorphisms of $\mathbb{P}^1$). (We will leave for the reader the analogous substitutions for the other solutions below.) Using the formulae in Appendix A one finds the family of systems parameterised by $s \in \mathbb{P}^1$ is (up to overall conjugation):

$$\frac{d}{dz} - \left(\frac{A_1}{z} + \frac{A_2}{z-t(s)} + \frac{A_3}{z-1}\right)$$

where

$$A_1 = \begin{pmatrix} (s^2+3)(s^6-51s^4+99s^2-81) & 4s(s^4-9), \\ 4(5s^6-75s^4+135s^2-81)s(s^4-9) & -(s^2+3)(s^6-51s^4+99s^2-81) \end{pmatrix} \Big/ \Delta,$$

$$A_2 = \begin{pmatrix} 4(s+3)(s-1)^2s^2(s^3-s^2+3s+9) & -2(s+3)(s-1)^2(s^2+2s+3) \\ -2(s+3)(s-1)^2(s^3-3s^2-9s-9)(5s^5-5s^4-45s-27) & -4s^2(s+3)(s-1)^2(s^3-s^2+3s+9) \end{pmatrix} \Big/ \Delta,$$

$$A_3 = \mathrm{diag}(-\theta_4, \theta_4)/2 - A_1 - A_2, \quad \Delta = -36(s^2+3)(s^2-1)^2(s^2-9).$$

Note that if the denominator Δ is zero then $t \in \{0, 1, \infty\}$ since

$$1 - t = 2\,\frac{(s^2+3)^2(s^2-3)}{(s+3)^3(s-1)^3}.$$

Thus the system is well defined for all s in $t^{-1}(\mathbb{P}^1\setminus\{0, 1, \infty\})$ except possibly at $s = \infty$ (where $t = -1$). However writing $s = 1/s'$ it is easy to conjugate the system to one well-defined also at $s = \infty$. Thus one never encounters configurations requiring a nontrivial bundle; the Malgrange divisor is trivial in this situation (in spite of the fact the solution y does have a pole at $s = \infty$); indeed one knows the corresponding τ function (whose zeros lying over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ correspond to nontrivial bundles) satisfies

$$d\log(\tau) = \mathrm{Tr}\left(A_2\left(\frac{A_1}{t} + \frac{A_3}{t-1}\right)\right)dt$$
$$= -\frac{s^6 + 6\,s^5 + 3\,s^4 - 8\,s^3 - 9\,s^2 - 54\,s - 27}{3\,(s^4 - 9)(s^2 - 1)(s^2 - 9)}\,ds$$

which is nonsingular for all $s \in t^{-1}(\mathbb{P}^1 \setminus \{0, 1, \infty\})$.

Remark 3. Sometimes one is interested in Fuchsian equations with given monodromy, rather than systems. To obtain these one may choose a cyclic vector, or more simply substitute the P_{VI} solution into the standard formulae for the isomonodromic family of Fuchsian equations. In the present case one obtains the equation

$$\frac{d^2}{dz^2} + a_1\frac{d}{dz} + a_2,$$

where a_1 and $z(z-1)a_2$ are respectively

$$\frac{1}{2z} + \frac{2}{3\,(z-1)} + \frac{2}{3(z - t(s))} - \frac{1}{z - y(s)},$$
$$\frac{7s^6 - 6s^5 + 3s^4 + 4s^3 - 63s^2 - 54s - 27}{18(s+3)^3(s-1)^3(z - t(s))} - \frac{(s^2 - 2s + 3)s^2}{9(s+3)(s-1)^2(z - y(s))} - \frac{1}{18}.$$

For generic s this is a Fuchsian equation with non-apparent singularities at $z = 0, 1, t, \infty$ and an apparent singularity at $z = y$, realising the given (projective) monodromy representation. In special cases (when $y = 0, 1, t, \infty$) it will have just the four non-apparent singularities (and will thus be a so-called “Heun equation”). For example specialising to $s = 0$ one finds $y = 0$, $t = -1$ and the equation becomes that with

$$a_1 = -\frac{1}{2z} + \frac{2}{3\,(z-1)} + \frac{2}{3(z+1)}, \quad a_2 = -\frac{1}{18\,(z-1)\,(z+1)}$$

which is a Heun equation whose projective monodromy representation is that specified by row 6 of Table 2.

Remark 4. At the editor’s request we will explain how one may verify directly that these P_{VI} solutions actually do correspond to Fuchsian systems with linear monodromy representations as specified by Table 2. For the rigid cases, rows 1 and 2, this is immediate, by rigidity. For the others, first one may check that the solutions actually do solve P_{VI}. This can be done directly (by computing the derivatives of y with respect to t and substituting into the P_{VI} equation).[2] Having done this we know the formulae of Appendix A do indeed give an isomonodromic family of Fuchsian systems. To see it has the monodromy representation specified by Table 2 we first compute the Puiseux expansions at 0 of each branch of the function $y(t)$ (only the leading terms will be needed). On the other hand, Jimbo's asymptotic formula (in the form in [7], Theorem 4) computes the leading term in the asymptotic expansion of the P_{VI} solution corresponding to the given monodromy representation ρ (the leading term is of the form at^b where a and b are explicit functions of $\theta_1, \theta_2, \theta_3, \theta_4, \sigma_{12}, \sigma_{23}, \sigma_{13}$). Then it is sufficient to check this leading term equals one of the leading terms of the Puiseux expansions of $y(t)$. The logic is that, in the cases at hand, the leading term determines the whole Puiseux expansion (using the recursion determined by the P_{VI} equation) and this is convergent so determines the solution locally, and thus globally by analytic continuation. (For the solutions we construct here this is automatic since we constructed the solution starting with the results of Jimbo's formula.)

Some simpler, but not entirely conclusive, checks are as follows:

1) Compare the monodromy of the Belyi map t with the $\mathcal{F}_2$ action (7) on the conjugacy class of the representation ρ (if we didn't know better it would appear as a miracle that the solution, constructed out of just the Puiseux expansion at 0, turns out to have the right branching at 1 and ∞ too).

2) Compute directly the Galois group of one of the Fuchsian systems in the isomonodromic family. (Together with the exponents this goes a long way to pinning down the monodromy representation.) There are various ways to do this, one of which is to convert the system into an equation (e.g. via a cyclic vector) and use the facility on Manuel Bronstein's webpage: http://www-sop.inria.fr/cafe/Manuel.Bronstein/sumit/bernina_demo.html (This requires finding a suitable rational point on the Painlevé curve, which, if possible, is easy in the genus zero cases, and not too difficult using Magma in the genus one cases.)

4 The octahedral solutions

For the octahedral group we do better and find more new solutions. In this case, by [11] or direct computation, S has size 3360, which reduces to just thirteen classes under either geometric or parameter equivalence. Thus there are exactly thirteen octahedral solutions to P_{VI}, up to equivalence under Okamoto's affine F_4 action.

Data about these classes are listed in Tables 3 and 4. In this case the type of the solution may contain the symbol "g" which indicates that one of the corresponding rotations in SO_3 is a rotation by a quarter of a turn. Also, in some cases rather than list the monodromy group of the cover $t: \Pi \to \mathbb{P}^1$ we just give its size.

[2] To aid the reader interested in examining the solutions of this article (and to help avoid typographical errors) a Maple text file of the solutions has been included with the source file of the preprint version on the math arxiv (math.DG/0501464). This may be downloaded by clicking on "Other formats" and unpacked with the commands 'gunzip 0501464.tar' and 'tar -xvf 0501464.tar', at least on a Unix system.

Table 3. Properties of the octahedral solutions.

	Deg.	Genus	Walls	Type	Alcove Point	n	Group	Partitions
1	1	0	1	abg	(65, 35, 25, 5)/2	192	1	
2	1	0	2	bg^2	25, 10, 10, 5	96	1	
3	2	0	2	b^2g^2	45, 15, 10, 10	96	S_2	$1, 2$
4	3	0	1	abg^2	40, 10, 5, 5	288	S_3	$3, 2$
5	4	0	2	ag^3	(75, 15, 15, 15)/2	128	A_4	3
6	4	0	3	g^4	30, 0, 0, 0	32	A_4	3
7	6	0	2	a^2bg	(95, 25, 5, 5)/2	576	24	$2^2, 3^2, 2\,4$
8	6	0	2	b^2g^2	35, 5, 0, 0	288	36	$3, 2\,4$
9	8	0	1	ab^2g	(85, 15, 15, 5)/2	768	576	$2^2\,3, 2^2\,4$
10	8	0	3	a^2g^2	45, 15, 0, 0	192	192	$3^2, 2\,3^2$
11	12	0	3	a^2b^2	50, 10, 0, 0	288	576	$2^2\,3^2, 2^2\,4^2$
12	12	1	3	a^3b	55, 5, 5, 5	288	96	$3^4, 2^2\,4^2$
13	16	0	3	a^3g	(105, 15, 15, 15)/2	128	3072	$2^2\,3^4$

The octahedral solutions with at most 4 branches correspond to the following known solutions. As in [5] one finds: The first two classes correspond to the octahedral entries on Schwarz's list of algebraic hypergeometric functions (and the P_{VI} solution is $y = t$ with the parameters indicated in Table 4). Solution 3 is $y = \pm\sqrt{t}$ with the parameters listed in Table 4, solution 4 has 3 branches and is a simple deformation of the 3-branch tetrahedral solution above (namely it is the solution in equation (8), but with the parameters given in Table 4), solution 5 is a simple deformation of the 4-branch dihedral solution (namely it is the solution in equation (9), but with the parameters given in Table 4), and solution 6 is the 4-branch octahedral solution

$$y = \frac{(s-1)^2}{s(s-2)}, \quad t = \frac{(s+1)(s-1)^3}{s^3(s-2)} \tag{10}$$

on p. 588 of [13], with the parameters as in Table 4, which is equivalent to a solution found independently by Dubrovin [9] (E.29).

For the remaining 7 solutions, rows 7–13, we will construct an explicit solution in each class using Jimbo's asymptotic formula. More computational details appear in Appendix C. (We have recently learnt that solutions 8 and 10 are equivalent to those of [18], 3.3.3 top of p. 22, and 3.3.5 bottom of p. 23, respectively.) The formulae obtained are as follows.

Octahedral solution 7, 6 branches, $(\theta_1, \theta_2, \theta_3, \theta_4) = (1/2, 1/2, 1/4, 2/3)$:

$$y = \frac{9s(2s^3 - 3s + 4)}{4(s+1)(s-1)^2(2s^2+6s+1)}, \quad t = \frac{27s^2}{4(s^2-1)^3}.$$

Table 4. Representative parameters for the octahedral solutions.

	$(\theta_1,\theta_2,\theta_3,\theta_4)$	$(\sigma_{12},\sigma_{23},\sigma_{13})$
1	1/2, 0, 1/3, 1/4	1/2, 1/3, 1/4
2	1/3, 0, 1/4, 1/4	1/3, 1/4, 1/4
3	1/3, 1/4, 1/4, 2/3	1/2, 1/2, 1/2
4	1/2, 1/4, 1/4, 2/3	1/2, 1/3, 3/4
5	1/4, 1/4, 1/4, 1/2	1/3, 1/2, 1/3
6	1/4, 1/4, 1/4, 1/4	1/3, 0, 1/3
7	1/2, 1/2, 1/4, 2/3	1/2, 1/2, 1/3
8	1/3, 3/4, 1/3, 3/4	1/2, 3/4, 1/3
9	1/3, 1/4, 1/2, 2/3	1/2, 2/3, 3/4
10	1/2, 1/4, 1/2, 3/4	2/3, 2/3, 1
11	1/3, 1/2, 1/2, 2/3	1/2, 1/2, 1/4
12	1/2, 1/2, 1/2, 2/3	1/2, 1/4, 2/3
13	1/2, 1/2, 1/2, 3/4	1/2, 2/3, 1/3

Octahedral solution 8, 6 branches, $(\theta_1,\theta_2,\theta_3,\theta_4)=(1/3,3/4,1/3,3/4)$:

$$y=\frac{(2s^2-1)(3s-1)}{2s(2s^2+2s-1)(s-1)},\qquad t=-\frac{(3s-1)^2}{8(2s^2+2s-1)(s-1)s^3}.$$

Octahedral solution 9, 8 branches, $(\theta_1,\theta_2,\theta_3,\theta_4)=(1/3,1/4,1/2,2/3)$:

$$y=\frac{s^3(2s^2-4s+3)(s^2-2s+2)}{(2s^2-2s+1)(3s^2-4s+2)},\qquad t=\left(\frac{s^2(2s^2-4s+3)}{3s^2-4s+2}\right)^2.$$

Octahedral solution 10, 8 branches, $(\theta_1,\theta_2,\theta_3,\theta_4)=(1/2,1/4,1/2,3/4)$:

$$y=\frac{32s(s+1)(5s^2+6s-3)}{(s^2+2s+5)(3s^2+2s+3)^2},\qquad t=\frac{1024s^3(s+1)^2}{(s^2+6s+1)(3s^2+2s+3)^3}.$$

Octahedral solution 11, 12 branches, $(\theta_1,\theta_2,\theta_3,\theta_4)=(1/3,1/2,1/2,2/3)$:

$$y=\frac{(s+1)(7s^4+16s^3+4s^2-4)\,r}{s^3(s-2)(s^4-4s^2+32s-28)},\qquad t=\left(\frac{(s+1)^2\,r}{(s-2)^2s^2}\right)^2$$

where $r=4(3s^2-4s+2)/(s^2+4s+6)$.

The next solution, number 12, has genus one. In this case we take Π to be the elliptic curve

$$u^2=(2s+1)(9s^2+2s+1).$$

As functions on this curve the solution is as follows.

Octahedral solution 12, genus one, 12 branches, $\theta = (1/2, 1/2, 1/2, 2/3)$:

$$y = \frac{1}{2} + \frac{45\,s^6 + 20\,s^5 + 95\,s^4 + 92\,s^3 + 39\,s^2 - 3}{4\,(5\,s^2 + 1)(s+1)^2 u},$$

$$t = \frac{1}{2} + \frac{s(2\,s+1)^2(27\,s^4 + 28\,s^3 + 26\,s^2 + 12\,s + 3)}{(s+1)^3 u^3}.$$

Finally the last solution, number 13, has 16 branches and genus zero. This is possibly the highest degree genus zero solution amongst all algebraic solutions of P_{VI}. It is also special since it has no real branches. Thus necessarily the parameterisation of the solution is not defined over $\mathbb{Q}$ although the solution curve $F(y,t) = 0$ itself has $\mathbb{Q}$ coefficients, as does the Belyi map t.

Octahedral solution 13, 16 branches, $(\theta_1, \theta_2, \theta_3, \theta_4) = (1/2, 1/2, 1/2, 3/4)$:

$$y = -\frac{(1+i)(s^2-1)(s^2+2\,is+1)(s^2-2\,is+1)^2 c}{4\,s(s^2+i)(s^2-i)^2(s^2+(1+i)s-i)d},$$

$$t = \frac{(s^2-1)^2(s^4+6\,s^2+1)^3}{32\,s^2(s^4+1)^3}.$$

Here c and d are respectively

$$s^8 - (2-2i)\,s^7 - (6+2i)\,s^6 + (10+2i)\,s^5 + 4\,is^4 + (10-2i)\,s^3 + (6-2i)\,s^2 - (2+2i)\,s - 1,$$

$$s^6 - (3+3i)\,s^5 + 3\,is^4 + (4-4i)\,s^3 + 3\,s^2 + (3+3i)\,s + i.$$

Remark 5. The author has recently understood that an alternative way to construct some (but not all) of these tetrahedral and octahedral solutions would have been to use the quadratic transformations of [19]. For example tetrahedral solution 6 could have been obtained from tetrahedral solution 4 in this way (a fact that was apparently not noticed in [1]). It is debatable whether this would have been simpler for us than the direct method used here, given what had already been done in [7], [5]. (The quadratic transformations were crucial however to construct the higher genus icosahedral solutions, cf. [6].)

5 Infinite monodromy groups

In this final section we will give some examples of solutions corresponding to non-rigid representations ρ into some infinite subgroups of $G = SL_2(\mathbb{C})$. The point is that the method we are using to construct P_{VI} solutions should work provided that the $\mathcal{F}_2$ orbit of ρ is finite, and above we just used the finiteness of the image of ρ as a convenient way of ensuring this.

Thus we are looking for representations ρ having finite $\mathcal{F}_2$ orbits, or more concretely, matrices $M_1, M_2, M_3 \in G$ having finite orbit under the action (7). (Such an $\mathcal{F}_2$ orbit, on conjugacy classes of representations, gives the permutation representation of the Belyi cover $t\colon \Pi \to \mathbb{P}^1$ for the corresponding $\mathrm{P_{VI}}$ solution. We would like to find interesting $\mathcal{F}_2$ orbits in order to find interesting $\mathrm{P_{VI}}$ solutions.) So far there appear to be four ways (apart from guessing) of finding representations ρ having finite $\mathcal{F}_2$ orbits.

Firstly one can just set the parameters to be sufficiently irrational in one of the families of solutions. (For example $y = \sqrt{t}$ is a solution provided $\theta_1 + \theta_4 = 1, \theta_2 = \theta_3$ amongst other possibilities.)

Secondly one can sometimes apply an Okamoto transformation to a known solution and change ρ into a representation having infinite image. For example if we take the 16 branch octahedral solution above and apply the Okamoto transformation corresponding to the central node of the extended D_4 Dynkin diagram, then we obtain a $\mathrm{P_{VI}}$ solution whose corresponding linear representation has image equal to the (2, 3, 8) triangle group. To see this we recall [14], [7] that Okamoto's affine D_4 action does not change the quadratic functions $\mathrm{Tr}(M_i M_j) = 2\cos(\pi\sigma_{ij})$ of the monodromy data, only the θ-parameters. In this case the 16 branch octahedral solution has data

$$\theta = (1/2, 1/2, 1/2, 3/4), \quad \sigma = (1/2, 2/3, 1/3)$$

on one branch and the solution after applying the transformation has data

$$\theta = (3/8, 3/8, 3/8, 5/8), \quad \sigma = (1/2, 2/3, 1/3).$$

One may show that the image of the corresponding triple (M_1, M_2, M_3) in $\mathrm{PSL}_2(\mathbb{C})$ generates a (2, 3, 8) triangle group (see appendix B). The corresponding solution to $\mathrm{P_{VI}}$ is given by the formula

$$y_{238}(s) = y(s) + \frac{2 - \sum_1^4 \theta_i}{2\, p(y, y', t)}$$

where y, t, θ_i are as for the 16 branch octahedral solution and p as in Appendix A. Explicitly one finds the solution is

2, 3, 8 solution, genus zero, 16 branches, $\theta = (3/8, 3/8, 3/8, 5/8)$:

$$y = -\frac{(1+i)(s^2-1)(s^2+2is+1)(s^2-2is+1)^2 d'}{8s(s^2+i)(s^2-i)^2 d}$$

with t and $d(s)$ as for the 16 branch octahedral solution, and $d'(s) = \overline{d(\overline{s})}$. In turn, via the formulae in Appendix A, this yields the explicit family of Fuchsian systems having projective monodromy group the (2, 3, 8) triangle group.

Thirdly, the idea of [7] was to use a different realisation of $\mathrm{P_{VI}}$ as controlling isomonodromic deformations of certain 3×3 systems. The corresponding monodromy representations were subgroups of $\mathrm{GL}_3(\mathbb{C})$ generated by complex reflections, and again one will obtain finite branching solutions by taking representations into a finite group generated by complex reflections. Applying this to the Klein complex reflection group led to an algebraic solution to P_{VI} with 7 branches. Moreover [7] described explicitly how to go between this 3×3 picture and the standard 2×2 picture used here, both on the level of systems and monodromy data. The upshot is that if we substitute the Klein solution into the formulae of appendix A below, then we obtain an isomonodromic family of 2×2 Fuchsian systems with monodromy data on one branch given by

$$\theta = (2/7, 2/7, 2/7, 4/7), \quad \sigma = (1/2, 1/3, 1/2). \tag{11}$$

This determines a representation into (a lift to G of) the (2, 3, 7) triangle group (and one may show as in Appendix B its image is not a proper subgroup). Moreover it was proved in [7] that this cannot be obtained by Okamoto transformations from a representation into a finite subgroup of $SL_2(\mathbb{C})$.

Fourthly one may obtain such representations by pulling back certain hypergeometric systems along certain rational maps (cf. Doran [8] and Kitaev [18]). This is closely related Klein's theorem that all second order Fuchsian equations with finite monodromy group are pull-backs of hypergeometric equations with finite monodromy. The basic idea is as follows.

Label two copies of $\mathbb{P}^1$ by u and d (for upstairs and downstairs). Choose four integers $n_0, n_1, n_\infty, N \geq 2$ and suppose we have an algebraic family of branched covers

$$\pi : \mathbb{P}^1_u \to \mathbb{P}^1_d$$

of degree N, parameterised by a curve Π say, such that:

1) π only branches at four points $0, 1, \infty$ and at a variable point $x \in \mathbb{P}^1_d$.

2) All but four of the ramification indices over $0, 1, \infty$ divide n_0, n_1, n_∞ respectively. In other words, if $\{e_{i,j}\}$ are the ramification indices over $i = 0, 1, \infty$, then, as j varies, precisely four of the numbers

$$\frac{e_{0j}}{n_0}, \quad \frac{e_{1j}}{n_1}, \quad \frac{e_{\infty j}}{n_\infty}$$

are not integers. Let t be the cross-ratio of the corresponding four ramification points of $\mathbb{P}^1_u$, in some order, and so we have a coordinate on $\mathbb{P}^1_u$ such that these four points occur at $0, t, 1, \infty$.

3) π has minimal ramification over x, i.e. π ramifies at just one point over x, with ramification index 2.

The idea of [8], [18] is to take a hypergeometric system on $\mathbb{P}^1_d$ with projective monodromy around i equal to an n_i'th root of the identity, for $i = 0, 1, \infty$ and pull it back along π. One then obtains an isomonodromic family on $\mathbb{P}^1_u$ with non-apparent singularities at $0, t, 1, \infty$ and possibly some apparent singularities at the other ramification points. All of the apparent singularities can be removed, for example by applying suitable Schlesinger transformations, yielding an isomonodromic family of systems of the desired form.

In particular the problem of constructing algebraic solutions of P_{VI} now largely becomes a purely algebraic problem about families of covers, although it is only conjectural that all algebraic solutions arise in this way.

However the algebraic construction of such covers seems difficult. First one has the topological problem of finding such covers, then one needs to find the full family of covers explicitly (this amounts to finding a parameterised solution of a large system of algebraic equations, typically with one less equation than the number of variables, so the solution is a curve). See [18] for some interesting examples however (but one should be aware that some of these solutions are equivalent to each other and to known solutions via Okamoto transformations).

Our perspective here is that just the topology of the cover is enough to determine the monodromy of the Fuchsian equation on $\mathbb{P}^1_u$ and we can then apply our previous method [7] to construct the explicit P_{VI} solution. In other words just one topological cover π gives us the desired representation ρ living in a finite $\mathcal{F}_2$ orbit.

To find some interesting topological covers we consider the list appearing in Corollary 4.6 of [8]. Here Doran classified the possible ramification indices of the cover π in the cases where the monodromy group of the hypergeometric system downstairs is an arithmetic triangle group in $SL_2(\mathbb{R})$. (Contrary to the wording in [8] this does not determine the topology of the cover.) We will content ourselves with looking at the last four entries of Doran's list, which say that the integers N, n_0, n_1, n_∞ and the ramification indices are

$$\begin{array}{llll} 10,\ 2,\ 3,\ 7, & [2,\ldots,2], & [3,3,3,1], & [7,1,1,1]; \\ 12,\ 2,\ 3,\ 7, & [2,\ldots,2], & [3,3,3,3], & [7,2,1,1,1]; \\ 12,\ 2,\ 3,\ 8, & [2,\ldots,2], & [3,3,3,3], & [8,1,1,1,1]; \\ 18,\ 2,\ 3,\ 7, & [2,\ldots,2], & [3,\ldots,3], & [7,7,1,1,1,1]. \end{array}$$

The basic problem now is to find such covers topologically, in other words to find the possible permutation representations. (The cover of the four punctured sphere $\mathbb{P}^1_d \setminus \{x, 0, 1, \infty\}$ is determined by its monodromy, which amounts to four elements of Sym_N having product equal to the identity and whose conjugacy classes – i.e. cycle types – are as specified by the given ramification indices.)

The simplest way to do this is to draw a picture. Suppose we fix $x = -1$ and cut $\mathbb{P}^1_d$ along the interval $I := [-1, \infty]$ from -1 along the positive real axis. Then the preimage of I under π will be a graph in $\mathbb{P}^1_u$ with vertices at each point of $\pi^{-1}(\{x, 0, 1, \infty\})$. The complement of the graph will be the union of exactly N connected components which are each mapped isomorphically by π onto $\mathbb{P}^1_d \setminus I$, and in particular the boundary of each component is the same as the boundary of $\mathbb{P}^1_d \setminus I$. These connected components correspond to the branches of π and the graph specifies how to glue them together. In particular the graph determines the permutation representation of the cover, since it shows us how to lift loops in the base to paths in $\mathbb{P}^1_u$; we just cross the corresponding edges upstairs, and note which connected component we end up in.

Thus we need to draw the graphs in $\mathbb{P}^1_u$. There are four types of vertices, depending on if they lie over $-1, 0, 1, \infty$, which we could draw as circles, squares, blobs and stars (say) respectively. The number of each type of vertices is just the number of points of $\mathbb{P}^1_u$ lying over the corresponding point amongst $-1, 0, 1, \infty$. The corresponding ramification indices give the number of edges coming out of each vertex to each of the neighbouring vertices, and our task is to join these edges together in a consistent manner.

For example for the first row of the above list, there should be 10 branches and, by examining the ramification indices, we see we need to draw a graph on $\mathbb{P}^1_u$ out of the following pieces:

- 8 circles with 1 edge emanating from each, and 1 circle with 2 edges,
- 5 squares with 4 edges,
- 3 blobs with 6 edges and one blob with 2 edges, and
- 1 star with 7 edges and 3 stars with 1 edge.

The graph should divide the sphere into 10 pieces. Furthermore:

- Each edge from a circle should connect to a square.
- Two edges from each square should connect to a circle and the other two should connect to a blob (and, going around the square, the edges should alternate between going to circles and blobs).
- Similarly half the edges from each blob should connect to squares and, again alternating, the other half should connect to stars.
- Finally each edge from a star should connect to one of the blobs.

We leave the reader to draw such a graph (there are 15 possibilities).[3] Given any such graph we can write down the monodromy of the pulled back Fuchsian system on $\mathbb{P}^1_u$ in terms of that of the hypergeometric system downstairs. Here the projective monodromy downstairs is a (2, 3, 7) triangle group

$$\Delta_{237} \cong \langle\, a, b, c \mid a^2 = b^3 = c^7 = cba = 1 \,\rangle$$

which can be realised as a subgroup of $\mathrm{PSL}_2(\mathbb{C})$ in various ways (the standard representation into $\mathrm{PSL}_2(\mathbb{R})$ plus its two Galois conjugates, lying in PSU_2).

Puncture $\mathbb{P}^1_u$ at the four exceptional vertices (namely the 3 stars with 1 edge and the blob with 2 edges) and choose generators $l_1, \ldots, l_4$ of the fundamental group of this punctured sphere, with $l_4 \circ \cdots \circ l_1$ contractible. Then we can compute the image under π of each l_i in $\mathbb{P}^1_d \setminus \{0, 1, \infty\}$ and thereby write the monodromy of the pulled back system as words in $a, b, c \in \Delta_{237}$. With one such graph we obtained

$$L_1 = caca^{-1}c^{-1}, \quad L_2 = c,$$
$$L_3 = c^{-1}a^{-1}cac, \quad L_4 = c^{-3}bc^3,$$

[3] To count the possibilities, one may use Theorem 7.2.1 in Serre's book [26] to count the number of such permutation representations, and then divide by conjugation action of the symmetric group, carefully computing the stabiliser. To find all possibilities we draw some and then apply the natural action of the pure three-string braid group to see if we get them all – here all 15 are braid equivalent.

where L_i is the projective monodromy around l_i. By construction $L_4 \dots L_1 = 1$ in Δ_{237}. Now by choosing an embedding of Δ_{237} in $\mathrm{PSL}_2(\mathbb{C})$ we get $L_i \in \mathrm{PSL}_2(\mathbb{C})$ and we can lift each L_i to a matrix $M_i \in G$, (and possibly negate M_4 to ensure $M_4 \dots M_1 = 1$). We obtain the representation ρ with data

$$\theta = (2/7, 2/7, 2/7, 1/3), \quad \sigma = (1/3, 1/3, 1/7).$$

This completes our task of producing a representation in a finite $\mathcal{F}_2$ orbit. Now we can apply our previous method to construct the corresponding $\mathrm{P_{VI}}$ solution. Immediately, by computing the $\mathcal{F}_2$ orbit of the conjugacy class of ρ, we find the solution has genus 1 and 18 branches, and that the parameters are not equivalent to those of any known solution. Moreover it turns out that Jimbo's asymptotic formula may be applied to 17 of the 18 branches, and the asymptotics on the remaining branch may be obtained by the lemma in section 7 of [5]. Using this we can get the solution polynomial F explicitly from the Puiseux expansions, and then look for a parameterisation of F. The result is as follows.

2, 3, 7 solution, genus one, 18 branches, $\theta = (2/7, 2/7, 2/7, 1/3)$:

$$y = \frac{1}{2} - \frac{(3\,s^8 - 2\,s^7 - 4\,s^6 - 204\,s^5 - 536\,s^4 - 1738\,s^3 - 5064\,s^2 - 4808\,s - 3199)u}{4\,(s^6 + 196\,s^3 + 189\,s^2 + 756\,s + 154)(s^2 + s + 7)(s + 1)},$$

$$t = \frac{1}{2} - \frac{(s^9 - 84\,s^6 - 378\,s^5 - 1512\,s^4 - 5208\,s^3 - 7236\,s^2 - 8127\,s - 784)u}{432\,s(s + 1)^2(s^2 + s + 7)^2} \tag{12}$$

where

$$u^2 = s\,(s^2 + s + 7).$$

This solution is noteworthy in that currently there is no known relation to a Fuchsian system with finite monodromy group (one might speculate as to the existence of another realisation of $\mathrm{P_{VI}}$ in which this solution corresponds to such a Fuchsian system, but this is unknown).

For the other three entries on the excerpt of Doran's list above, we do not seem to get new solutions, but it is interesting to identify them in any case.

The second entry, a family of degree 12 covers, turns out to give the Klein solution of [7]. The explicit family of covers has been found more recently in [18], p. 27. There are 7 different graphs one could draw, one of which is symmetrical and they are all braid equivalent. Using one of these graphs we obtain, as above, the words

$$L_1 = c^{-1}a^{-1}cac, \quad L_2 = c^3aca^{-1}c^{-3},$$
$$L_3 = c^2aca^{-1}c^{-2}, \quad L_4 = (aca)^{-1}c^2aca.$$

Choosing an appropriate embedding of Δ_{237} and lifting to G we obtain the representation ρ specified in (11). In particular this gives a convenient way to prove that the projective monodromy group of the family of 2×2 Fuchsian systems determined by the Klein solution is Δ_{237}. We just need to show that the L_i generate all of Δ_{237}, which we will do in Appendix B below.

The third entry indicates a family of degree 12 covers along which one should pull back the (2, 3, 8) triangle group. This time there are 7 graphs one could draw but they are not all braid equivalent, there are two P_3 orbits, distinguished by the fact that the monodromy group of the cover is either Sym_{12} or a group of order 1536. For the degenerate case one finds the P_{VI} solution has just two branches and is $y = t \pm \sqrt{t(t-1)}$ with parameters $\theta = (1, 1, 1, 7)/8$. (This is just the transform of the square root solution $y = \sqrt{t}$ under the Okamoto transformation $(y, t) \mapsto \left(\frac{y-t}{1-t}, \frac{t}{t-1}\right)$). The other case is more interesting; for one graph we obtain

$$L_1 = aca^{-1}, \qquad L_2 = c^{-2}a^{-1}cac^2,$$
$$L_3 = caca^{-1}c^{-1}, \quad L_4 = a^{-1}caca^{-1}c^{-1}a,$$

where now a, b, c generate the (2, 3, 8) triangle group:

$$\Delta_{238} \cong \langle\, a, b, c \mid a^2 = b^3 = c^8 = cba = 1 \,\rangle.$$

Now we can choose an embedding of Δ_{238} into $\mathrm{PSL}_2(\mathbb{C})$ and a lift to G (negating M_4 if necessary) such that we obtain the representation ρ with data

$$\theta = (3/8, 3/8, 3/8, 5/8), \quad \sigma = (1/2, 2/3, 1/3).$$

This is precisely that obtained above by applying an Okamoto transformation to the 16 branch octahedral solution (and gives a convenient way to prove, in Appendix B, that the projective monodromy group is Δ_{238}).

Finally there are 9 graphs corresponding to the last row of Doran's list, all braid equivalent. Even though the graphs are the most complicated in this case (and there is a quite attractive one with 4-fold symmetry), this case leads again to the 2-branch P_{VI} solution $y = t \pm \sqrt{t(t-1)}$ with the parameters $\theta = (1, 1, 1, 6)/7$ (and $\sigma = (1/2, 1/2, 5/7)$ on one branch).

In conclusion we should mention that we do not know any other finite $\mathcal{F}_2$ orbits of triples of elements of $\mathrm{SL}_2(\mathbb{C})$ (e.g. up to isomorphism as 'sets with $\mathcal{F}_2$-action'); so far they all either come from a finite subgroup or one of the two (2, 3, 7) cases (the Klein solution or the genus one solution above).

Appendix A

Here are the explicit formula from [16] for the residue matrices A_i, of the isomonodromic family of Fuchsian systems corresponding to a P_{VI} solution $y(t)$ with parameters $\theta_1, \dots, \theta_4$. The matrix entries are rational functions of $y, t, y' = \frac{dy}{dt}, \{\theta_i\}$. (Our coordinate x is denoted $\widetilde{z}$ in [16] and is related simply to p which is the usual dual variable to $y = q$ in the Hamiltonian formulation of P_{VI}. Also one should add $\mathrm{diag}(\theta_i, \theta_i)/2$ to our A_i to obtain that of [16].)

$$A_i := \begin{pmatrix} z_i + \theta_i/2 & -u_i z_i \\ (z_i + \theta_i)/u_i & -z_i - \theta_i/2 \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C})$$

where

$$z_1 = y\frac{E - k_2^2(t+1)}{t\theta_4}, \quad z_2 = (y-t)\frac{E + t\theta_4(y-1)xk_2^2 - tk_1k_2}{t(t-1)\theta_4},$$

$$z_3 = -(y-1)\frac{E + \theta_4(y-t)x - k_2^2 t - k_1k_2}{(t-1)\theta_4},$$

$$x = p - \frac{\theta_1}{y} - \frac{\theta_2}{y-t} - \frac{\theta_3}{y-1}, \quad 2p = \frac{\theta_1 + (t-1)\,y'}{y} + \frac{\theta_2 - 1 + y'}{y-t} + \frac{\theta_3 - t\,y'}{y-1},$$

$$u_1 = \frac{y}{tz_1}, \quad u_2 = \frac{y-t}{t(t-1)z_2}, \quad u_3 = -\frac{y-1}{(t-1)z_3},$$

$$E = y(y-1)(y-t)x^2 + \big(\theta_3(y-t) + t\theta_2(y-1) - 2k_2(y-1)(y-t)\big)x + k_2^2 y - k_2(\theta_3 + t\theta_2),$$

$$k_1 = (\theta_4 - \theta_1 - \theta_2 - \theta_3)/2, \quad k_2 = (-\theta_4 - \theta_1 - \theta_2 - \theta_3)/2.$$

Appendix B

Proposition 6. *The* 2×2 *Fuchsian systems corresponding to the Klein solution and to the* 18 *branch genus* 1 *solution of Section* 5 *have projective monodromy group isomorphic to* Δ_{237}*, and that corresponding to the transformation of the* 16 *branch octahedral appearing in Section* 5 *has projective monodromy group isomorphic to* Δ_{238}.

Proof. Since in Section 5 the projective monodromy groups were expressed as words in the generators of the respective triangle groups, it is sufficient to check in each case that these words are in fact generators. To do this we will repeatedly use the fact that in the group

$$\Delta = \langle\, a, b, c \mid a^2 = b^3 = c^n = cba = 1 \,\rangle$$

one has $c^b = bc^{-1}$ and $c^{b^{-1}} = c^{-1}b$, where in general we write x^y for $y^{-1}xy$. (These are easily verified, for example the first is true since

$$b^{-1}cbc = b^{-1}(cb)(cb)b^{-1} = b^{-2} = b,$$

using the fact that $a = a^{-1} = cb$.) In particular we immediately deduce $\Delta = \langle c, c^b \rangle = \langle c, c^{b^{-1}} \rangle$.

Now each case is an easy exercise. For the Klein case we need to show that $\langle L_i,\ i = 1, 2, 3\rangle = \Delta$ where $n = 7$ and

$$L_1 = c^{ac}, \quad L_2 = c^{ac^{-3}}, \quad L_3 = c^{ac^{-2}}.$$

Up to conjugacy in Δ, we have $\langle L_2, L_3\rangle = \langle p, c\rangle$ where $p = c^{ac^{-1}a}$. However, using $a = cb$ we see $p = c^{cb^2} = c^{b^2} = c^{b^{-1}}$ so we are done.

For the other (2, 3, 7) case corresponding to the genus one solution it is clear that $\langle L_2, L_4\rangle = \Delta$.

Finally for the (2, 3, 8) case we have

$$L_1 = c^a, \quad L_2 = c^{ac^2}, \quad L_3 = c^{ac^{-1}}.$$

Up to conjugacy $\langle L_1, L_3\rangle = \langle c, c^{ac^{-1}a}\rangle$ and as in the Klein case above $c^{ac^{-1}a} = bcb^{-1}$. □

Appendix C

At the request of the editors and of A. Kitaev, we will add some remarks to aid the reader interested in reproducing the results of this article. The main results are of two types: 1) classification of P_{VI} solutions coming from the binary tetrahedral and octahedral groups and 2) construction of explicit P_{VI} solutions using Jimbo's asymptotic formula. For both 1) and 2) the details are parallel to those described in [5] and [7] resp., with the precise references as in the body of this article. For 1) there are 3 steps:

• Prove that the relation of Okamoto equivalence is sandwiched between the relations of geometric and parametric equivalence, i.e. in symbols one has GE $\Rightarrow$ OE $\Rightarrow$ PE. The second arrow is immediate by definition and the first is proved in Lemma 9 of [5].

• Compute the parameter equivalence classes in the set of parameters coming from triples of generators of either the tetrahedral or octahedral group. This is as in section 3 of [5]. One first writes down the set of possible parameters θ. This is a finite subset of $\mathbb{Q}^4 \subset \mathbb{R}^4$. Then one uses a simple algorithm to move each of these points into a chosen affine F_4 alcove, using the standard action of the affine Weyl group $W_a(F_4)$ on $\mathbb{R}^4$ (this is entirely standard and the details are written in [5], Proposition 6). Then we count the number of distinct alcove points obtained. By definition this is the number of "parameter equivalence classes".

• Compute the geometric equivalence classes in the set of linear representations ρ coming from either the binary tetrahedral or octahedral group. This amounts to computing the orbits of an explicit action of a group on a finite set (of size 520, 3360 resp.) and is carefully described in section 4 of [5].

Some confidence that there is no computational error comes from the fact that the geometric and parametric equivalence classes turn out to coincide in both the cases considered here. Also Hall's formulae [11] (computing the number of generating triples) gives confidence that all the generating triples have been computed correctly – since we get the right number of them. (In principle one can go through all triples of elements of the finite group $\Gamma \subset SL_2(\mathbb{C})$ and throw out those that do not generate Γ.

In the two cases at hand this is feasible, but some simple tricks are useful in the icosahedral case.)

Now we will move on to 2), constructing the solutions. The main steps of the procedure used are as in [7] (see especially p. 193). However with experience various tricks have been developed to speed up the computation, so we will also detail some of these below (they are inessential if one has a fast enough computer, as presumably future readers will have). The underlying strategy is analogous to that used in [10] although we do not in fact use any of their results. (It was particularly troublesome to get the correct form of Jimbo's formula in [7], which is the main ingredient and was not used in [10].)

The basic steps are as follows:

1) We start with a linear representation ρ living in a finite mapping class group orbit. The conjugacy class of ρ is encoded in the seven-tuple

$$\theta_1,\ \theta_2,\ \theta_3,\ \theta_4,\ \sigma_{12},\ \sigma_{23},\ \sigma_{13}.$$

Specifying these seven numbers is equivalent to specifying the numbers $m_i = 2\cos(\pi\theta_i)$, $m_{ij} = 2\cos(\pi\sigma_{ij})$ provided we agree $\theta_i, \sigma_{ij} \in [0, 1]$. We compute the orbit of this 7-tuple under the pure mapping class group $\cong \mathcal{F}_2$. The formula for this action is given in [5], section 4 (cf. also (7) above). This gives a list, of length N say, of 7-tuples, one for each branch of the corresponding $\mathrm{P_{VI}}$ solution. The values of the θ's will not vary on different branches so the branches are parameterised by the values of the sigmas. Let their values on the kth branch be denoted σ_{ij}^k, $k = 1, \ldots, N$.

2) Plug each 7-tuple into Jimbo's asymptotic formula (in the form in [7], Theorem 4). This gives N leading terms $y_k = a_k t^{b_k} + \cdots$ for $k = 1, \ldots, N$ of the Puiseux expansion at 0 of the $\mathrm{P_{VI}}$ solution $y(t)$ on the N branches. One will have $b_k = 1 - \sigma_{12}^k$ but a_k is given by a very complicated, but explicit, formula. (Jimbo's formula is not always applicable – cf. the discussion of 'good' solutions in [5], but often there is an equivalent solution for which Jimbo's formula can be applied on every branch, or there is a degeneration of Jimbo's formula (as in [10] or [5], Lemma 19) which will compute the remaining leading terms.)

3) Compute lots of terms in the Puiseux expansions of the solutions $y(t)$ on each branch. These will be expansions in $s = t^{1/d_k}$ where d_k is the denominator of b_k (when written in lowest terms). Geometrically d_k is the number of branches of y that meet the given branch over $t = 0$, i.e. it is the cycle length of the given branch in the permutation representation of the solution curve as a cover of the t-line. The expansions are computed recursively by substituting back into the $\mathrm{P_{VI}}$ equation; at each step this leads to a linear equation for the coefficient of the next term in the expansion.

4) Use these expansions to determine the coefficients of the solution polynomial $F(y, t)$. (This determines y as an algebraic function of t by the condition $F(y, t) = 0$.) Since F is a polynomial (of degree N in y) this is clearly possible since we have arbitrarily many terms of each Puiseux expansion; $F(y_k(s), s^{d_k}) = 0$ for all s and for each branch y_k of the solution. (Thus in principle just one expansion is needed, not

the expansion for all branches.) Given $F(y,t)$ one may check symbolically that it specifies a solution to P_{VI}, using implicit differentiation.

5) Compute a parameterisation of the resulting curve $F(y,t)=0$. (As mentioned in the acknowledgments the author is grateful to Mark van Hoeij for help with this last step.) In general this will be simpler to write down than the polynomial F.

Now we will list some of the tricks we have found useful in carrying out the above steps.

1) One needs to convert the numbers a_k given by Jimbo's formula into algebraic numbers. In the examples so far this can be done by raising a_k (and/or its real/imaginary parts) to the d_k-th power until a rational number is obtained (which can be ascertained by looking at continued fraction expansions).

2) Reduce the number of Puiseux expansions to compute: the d_k branches which meet the given branch over zero will have Galois conjugate expansions. These can be obtained from one another by multiplying s by a d_k-th root of unity. Also, when choosing which of these d_k expansions to actually compute it is good to choose the real one, if possible. (Also sometimes some expansions are complex conjugate to others so further optimisations are possible.)

3) Reduce the degree of the field extension used to compute the expansion: In computing the Puiseux expansions one is often working over a finite extension of $\mathbb{Q}$, such as $\mathbb{Q}(6^{1/7})$. Often the degree of this extension can be reduced by taking the expansion in a variable $h=c\times s$ for a suitable constant c, rather than in $s=t^{1/d_k}$. This trick was very useful for computing the larger solutions (with ≥ 15 branches say).

4) To obtain the coefficients of the polynomial F from the Puiseux expansion we use the trick suggested in [10]: Write F in the form

$$F=q(t)y^N+p_{N-1}(t)y^{N-1}+\cdots+p_1(t)y+p_0(t)$$

for polynomials p_i, q in t and define rational functions $r_i(t):=p_i/q$ for $i=0,\ldots,N-1$. If $y_1,\ldots,y_N$ denote the (locally defined) solutions on the branches then for each t we have that $y_1(t),\ldots,y_N(t)$ are the roots of $F(t,y)=0$ and it follows that

$$y^N+r_{N-1}(t)y^{N-1}+\cdots+r_1(t)y+r_0(t)=(y-y_1(t))(y-y_2(t))\ldots(y-y_N(t)).$$

Thus, expanding the product on the right, the rational functions r_i are obtained as symmetric polynomials in the y_i:

$$r_0=(-1)^N y_1\ldots y_N,\ \ldots,\ r_{N-1}=-(y_1+\cdots+y_N).$$

Since the r_i are global rational functions, the Puiseux expansions of the y_i give the Laurent expansions at 0 of the r_i. Clearly only a finite number of terms of each Laurent expansion are required to determine each r_i, and indeed it is simple to convert these truncated Laurent expansions into global rational functions. (This is easily done by Padé approximation, e.g. as implemented in the Maple command "convert(, ratpoly)".)

5) Much time may be saved by carefully choosing the representative for the solution in the first place (i.e. try to choose an equivalent solution for which the polynomial F is simpler). Heuristically this can be estimated by seeing how complicated the algebraic numbers a_k are (or by seeing how complicated the coefficients of the polynomial $q(t)$ are; this is usually easily obtained from $(y_1 + \cdots + y_N)$, i.e. before having to compute complicated symmetric functions).

6) Use Okamoto symmetries wherever possible: e.g. if (we can arrange that) the solution has the symmetry $(y, t) \mapsto (y/t, 1/t)$, swapping θ_2 and θ_3 then the coefficients of each p_i, q should be symmetrical, thereby essentially halving the number of coefficients that need to be computed. (Also for the 24-branch icosahedral solution in [5] it was too cumbersome to compute the longest symmetric functions, corresponding to the 'middle' polynomials p_i, but, by using another Okamoto symmetry, the outstanding coefficients could be determined in terms of those we were able to compute.)

7) Finally there are various optimisations that can be made (especially in computing the symmetric functions of the Puiseux expansions) if we expect F to have integer coefficients (which is the case for all examples so far).

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After this work was complete A. Kitaev informed the author that he had found an explicit family of covers corresponding to the genus one (2, 3, 7) solution of Section 5 and had obtained a similar solution. Happily the solution here and that of Kitaev are not related by Okamoto transformations, but arise by choosing different embeddings of Δ_{237} into $\mathrm{PSL}_2(\mathbb{C})$. In fact there are three inequivalent choices, corresponding to the three conjugacy classes of order 7 elements in $\mathrm{PSL}_2(\mathbb{C}) \cong \mathrm{SO}_3(\mathbb{C})$. (This is analogous to the sibling solutions of [5] which arose from the two classes of order 5 elements.) The third inequivalent $\mathrm{P_{VI}}$ solution is:

$$y = \frac{1}{2} - \frac{(s^{10} + 5s^9 + 24s^8 + 20s^7 - 266s^6 - 2874s^5 - 14812s^4 - 40316s^3 - 85359s^2 - 100067s - 67396)u}{16(s+1)(s^2+s+7)(5s^6 + 63s^5 + 252s^4 + 854s^3 + 1449s^2 + 1827s + 2030)}$$

with t, u, s exactly as in (12) and $\theta = (4/7, 4/7, 4/7, 1/3)$.

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Galois theory of parameterized differential equations and linear differential algebraic groups

*Phyllis J. Cassidy and Michael F. Singer**

Department of Mathematics
The City College of New York
New York, New York 10038, U.S.A.
email: `pcassidy@ccny.cuny.edu`

Department of Mathematics
North Carolina State University, Box 8205
Raleigh, North Carolina 27695-8205, U.S.A.
email: `singer@math.ncsu.edu`

Abstract. We present a Galois theory of parameterized linear differential equations where the Galois groups are linear differential algebraic groups, that is, groups of matrices whose entries are functions of the parameters and satisfy a set of differential equations with respect to these parameters. We present the basic constructions and results, give examples, discuss how isomonodromic families fit into this theory and show how results from the theory of linear differential algebraic groups may be used to classify systems of second order linear differential equations.

Contents

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1 Introduction

We will describe a Galois theory of differential equations of the form

$$\frac{\partial Y}{\partial x} = A(x, t_1, \ldots, t_n)Y$$

where $A(x, t_1, \ldots, t_n)$ is an $m \times m$ matrix with entries that are functions of the principal variable x and parameters $t_1, \ldots, t_n$. The Galois groups in this theory are linear differential algebraic groups, that is, groups of $m \times m$ matrices $(f_{i,j}(t_1, \ldots, t_n))$ whose entries satisfy a fixed set of differential equations. For example, in this theory, the equation

$$\frac{\partial y}{\partial x} = \frac{t}{x} y$$

has Galois group

$$G = \left\{(f(t)) \mid f \neq 0 \text{ and } f\frac{d^2 f}{dt^2} - \left(\frac{df}{dt}\right)^2 = 0\right\}.$$

In the process, we will give an introduction to the theory of linear differential algebraic groups and show how one can use properties of the structure of these groups to deduce results concerning parameterized linear differential equations.

Various differential Galois theories now exist that go beyond the eponymous theory of linear differential equations pioneered by Picard and Vessiot at the end of the 19th century and made rigorous and expanded by Kolchin in the middle of the 20th century. These include theories developed by B. Malgrange, A. Pillay, H. Umemura and one presently being developed by P. Landesman. In many ways the Galois theory presented here is a special case of the results of Pillay and Landesman yet we hope that the explicit nature of our presentation and the applications we give justify our exposition. We will give a comparison with these theories in the final comments.

The rest of the paper is organized as follows. In Section 2 we review the Picard–Vessiot theory of integrable systems of linear partial differential equations. In Section 3 we introduce and give the basic definitions and results for the Galois theory of parameterized linear differential equations ending with a statement of the Fundamental Theorem of this Galois theory as well as a characterization of parameterized equations that are solvable in terms of parameterized liouvillian functions. In Section 4 we describe the basic results concerning linear differential algebraic groups and give many examples. In Section 5 we show that, in the regular singular case, isomonodromic families of linear differential equations are precisely the parameterized linear differential equations whose parameterized Galois theory reduces to the usual Picard–Vessiot theory. In Section 6 we apply a classification of 2×2 linear differential algebraic groups to show that any parameterized system of linear differential equations with regular singular points is equivalent to a system that is generic (in a suitable sense) or isomonodromic or solvable in terms of parameterized liouvillian functions. Section 7 gives two simple examples illustrating the subtleties of the inverse problem in our setting. In Section 8 we discuss the relationship between the theory presented here and other differential Galois theories and give some directions for future research. The Appendices contain proofs of the results of Section 3.

2 Review of Picard–Vessiot theory

In the usual Galois theory of polynomial equations, the Galois group is the collection of transformations of the roots that preserve all algebraic relations among these roots. To be more formal, given a field k and a polynomial $p(y)$ with coefficients in k, one forms the *splitting field* K of $p(y)$ by adjoining all the roots of $p(y)$ to k. The *Galois group* is then the group of all automorphisms of K that leave each element of k fixed. The structure of the Galois group is well known to reflect the algebraic properties of the roots of $p(y)$. In this section we will review the Galois theory of linear differential equations. Proofs (and other references) can be found in [40].

One can proceed in an analogous fashion with integrable systems of linear differential equations and define a Galois group that is a collection of transformations of solutions of a linear differential system that preserve all the algebraic relations among the solutions *and their derivatives*. Let k be a differential field[1], that is, a field k together with a set of commuting derivations $\Delta = \{\partial_1, \ldots, \partial_m\}$. To emphasize the role of Δ, we shall refer to such a field as a Δ*-field*. Examples of such fields are the field $\mathbb{C}(x_1, \ldots, x_m)$ of rational functions in m variables, the quotient field $\mathbb{C}((x_1, \ldots, x_m))$ of the ring of formal power series in m variables and the quotient field $\mathbb{C}(\{x_1, \ldots, x_m\})$ of the ring of convergent power series, all with the derivations $\Delta = \left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}\right\}$. If k is a Δ-field and $\Delta' \subset \Delta$, the field $C_k^{\Delta'} = \{c \in k \mid \partial c = 0 \text{ for all } \partial \in \Delta'\}$ is called the subfield of Δ'-constants of k. When $\Delta' = \Delta$ we shall write C_k for C_k^{Δ} and refer to this latter field as the field of constants of k. An *integrable system of linear differential equations* is a set of equations

$$\begin{aligned} \partial_1 Y &= A_1 Y \\ \partial_2 Y &= A_2 Y \\ &\vdots \\ \partial_m Y &= A_m Y \end{aligned} \tag{2.1}$$

where the $A_i \in \mathrm{gl}_n(k)$, the set of $n \times n$ matrices with entries in k, such that

$$\partial_i A_j - \partial_j A_i = [A_i, A_j] \tag{2.2}$$

for all i, j. These latter equations are referred to as the *integrability conditions*. Note that if $m = 1$, these conditions are trivially satisfied.

[1] All fields in this paper will be of characteristic zero

The role of a splitting field is assumed by the *Picard–Vessiot extension* associated with the integrable system (2.1). This is a Δ-extension field $K = k(z_{1,1}, \dots, z_{n,n})$ where

(1) the $z_{i,j}$ are entries of a matrix $Z \in \mathrm{GL}_n(K)$ satisfying $\partial_i Z = A_i Z$ for $i = 1, \dots, m$, and

(2) $C_K = C_k = C$, *i.e.,* the Δ-constants of K coincide with the Δ-constants of k.

Note that condition (1) defines uniquely the actions on K of the derivations ∂_i and that the integrability conditions (2.2) must be satisfied since these derivations commute. We refer to the Z above as a *fundamental solution matrix* and we shall denote K by $k(Z)$. If $k = \mathbb{C}(x_1, \dots, x_m)$ with the obvious derivations, one can easily show the existence of Picard–Vessiot extensions. If we let $\vec{a} = (a_1, \dots, a_n)$ be a point of $\mathbb{C}^n$ where the denominators of all entries of the A_i are holomorphic, then the Frobenius Theorem ([55],Ch. 1.3) implies that, in a neighborhood of $\vec{a}$, there exist n linearly independent analytic solutions $(z_{1,1}, \dots, z_{n,1})^T, \dots, (z_{1,n}, \dots, z_{n,n})^T$ of the equations (2.1). The field $k(z_{1,1}, \dots, z_{n,n})$ with the obvious derivations satisfies the conditions defining a Picard–Vessiot extension. In general, if k is an arbitrary Δ-field with C_k algebraically closed, then there always exists a Picard–Vessiot extension K for the integrable system (2.1) and K is unique up to k-differential isomorphism. We shall refer to K as the *PV-extension associated with* (2.1).

Let K be a PV-extension associated with (2.1) and let $K = k(Z)$ with Z a fundamental solution matrix. If U is another fundamental solution matrix then an easy calculation shows that $\partial_i(U^{-1}Z) = 0$ for all i and so $U^{-1}Z \in \mathrm{GL}_n(C_k)$. We define the Δ-*Galois group* $\mathrm{Gal}_\Delta(K/k)$ of K over k (or of the system (2.1)) to be

$$\mathrm{Gal}_\Delta(K/k) = \{\sigma : K \to K \mid \sigma \text{ is a } k\text{-automorphism of } K \text{ and } \\ \partial_i \sigma = \sigma \partial_i, \text{ for } i = 1, \dots, m\}.$$

Note that a k-automorphism σ of K such that $\partial_i \sigma = \sigma \partial_i$ is called a *k-differential automorphism*. For any $\sigma \in \mathrm{Gal}_\Delta(K/k)$, we have that $\sigma(Z)$ is again a fundamental solution matrix so the above discussion implies that $\sigma(Z) = ZA_\sigma$ for some $A_\sigma \in \mathrm{GL}_n(C_k)$. This yields a representation $\mathrm{Gal}_\Delta(K/k) \to \mathrm{GL}_n(C_k)$. Note that different fundamental solution matrices yield conjugate representations. A fundamental fact is that the image of $\mathrm{Gal}_\Delta(K/k)$ in $\mathrm{GL}_n(C_k)$ is Zariski-closed, that is, it is defined by a set of polynomial equations involving the entries of the matrices and so has the structure of an linear algebraic group. If G is a linear algebraic group defined over F (that is, defined by equations having coefficients in the field F) and E is any field containing F, we will use the notation $G(E)$ to denote the set of points of G having entries in E.

These facts lead to a rich Galois theory, originally due to E. Picard and E. Vessiot and given rigor and greatly expanded by E. R. Kolchin. We summarize the fundamental result in the following

Theorem 2.1. *Let k be a Δ-field with algebraically closed field of constants C and* (2.1) *be an integrable system of linear differential equations over k.*

(1) *There exists a PV-extension K of k associated with* (2.1) *and this extension is unique up to a differential k-isomorphism.*

(2) *The Δ-Galois group* $\mathrm{Gal}_\Delta(K/k)$ *may be identified with $G(C)$, where G is a linear algebraic group defined over C.*

(3) *The map that sends any differential subfield $F, k \subset F \subset K$, to the group* $\mathrm{Gal}_\Delta(K/F)$ *is a bijection between the set of differential subfields of K containing k and the set of algebraic subgroups of* $\mathrm{Gal}_\Delta(K/k)$*. Its inverse is given by the map that sends a Zariski closed group H to the field $K^H = \{z \in K \mid \sigma(z) = z$ for all $\sigma \in H\}$.*

(4) *A Zariski-closed subgroup H of* $\mathrm{Gal}_\Delta(K/k)$ *is a normal subgroup of* $\mathrm{Gal}_\Delta(K/k)$ *if and only if the field K^H is left set-wise invariant by* $\mathrm{Gal}_\Delta(K/k)$*. If this is the case, the map* $\mathrm{Gal}_\Delta(K/k) \to \mathrm{Gal}_\Delta(K^H/k)$ *is surjective with kernel H and K^H is a PV-extension of k with PV-group isomorphic to* $\mathrm{Gal}_\Delta(K/k)/H$*. Conversely, if F is a differential subfield of K containing k and F is a PV-extension of k, then* $\mathrm{Gal}_\Delta(F/K)$ *is a normal subgroup of* $\mathrm{Gal}_\Delta(K/k)$*.*

Remarks 2.2. 1. The assumption that C is algebraically closed is necessary for the existence of PV-extensions (*cf.,* [42]) as well as necessary to guarantee that there are enough automorphisms so that (3) is correct. Kolchin's development in [21] of the Galois correspondence for PV-extensions does not make this assumption and he replaced automorphisms of the PV-extension with embeddings of the PV-extension into a large field (a *universal differential field*). Another approach to studying linear differential equations over fields whose constants are not algebraically closed is given in [1] (see in particular Corollaire 3.4.2.4 and Exemple 3.4.2.6). One can also study linear differential equations over fields whose fields of constants are not algebraically closed using descent techniques (see [17]).

2. Theorem 2.1 is usually stated and proven for the case when $m = 1$, the ordinary differential case, although it is proven in this generality in [21]. The usual proofs in the ordinary differential case do however usually generalize to this case as well. In the appendix of [40], the authors also discuss the case of $m > 1$ and show how the Galois theory may be developed in this case. We will give a proof of a more general theorem in the appendix from which Theorem 2.1 follows as well.

3. Theorem 2.1 is a manifestation of a deeper result. If $K = k(Z)$ is a PV-extension then the ring $k\big[Z, \frac{1}{\det Z}\big]$ is the coordinate ring of a torsor (principal homogeneous space) V defined over k for the group $\mathrm{Gal}_\Delta(K/k)$, that is, there is a morphism $V \times G \to V$ denoted by $(v, g) \mapsto vg$ and defined over k such that $v1 = v$ and $(vg_1)g_2 = v(g_1g_2)$ and such that the morphism $V \times G \to V \times V$ given by $(v, g) \mapsto (v, vg)$ is an isomorphism. The path to the Galois theory given by first establishing this fact is presented in [15], [25] and [40] (although Kolchin was well aware of this fact as well, *cf.,* [21], Ch. VI.8 and the references there to the original papers.) This approach allows one to give an intrinsic definition of the linear algebraic group structure on the Galois group as well.

We end this section with a simple example that will also illuminate the Galois theory of parameterized equations.

Example 2.3. Let $k = \mathbb{C}(x)$ be the ordinary differential field with derivation $\frac{d}{dx}$ and consider the differential equation

$$\frac{dy}{dx} = \frac{t}{x}y$$

where $t \in \mathbb{C}$. The associated Picard–Vessiot extension is $k(x^t)$. The Galois group will be identified with a Zariski-closed subgroup of $\mathrm{GL}_1(\mathbb{C})$. When $t \in \mathbb{Q}$, one has that x^t is an algebraic function and when $t \notin \mathbb{Q}$, x^t is transcendental. It is therefore not surprising that one can show that

$$\mathrm{Gal}_\Delta(K/k) = \begin{cases} \mathbb{C}^* = \mathrm{GL}_1(\mathbb{C}) & \text{if } t \notin \mathbb{Q}, \\ \mathbb{Z}/q\mathbb{Z} & \text{if } t = \frac{p}{q}, (p,q) = 1. \end{cases} \qquad \square$$

3 Parameterized Picard–Vessiot theory

In this section we will consider differential equations of the form

$$\frac{\partial Y}{\partial x} = A(x, t_1, \ldots, t_m)Y$$

where A is an $n \times n$ matrix whose entries are functions of x and parameters $t_1, \ldots, t_m$ and we will define a Galois group of transformations that preserves the algebraic relations among a set of solutions and their derivatives *with respect to all the variables*. Before we make things precise, let us consider an example.

Example 3.1. Let $k = \mathbb{C}(x,t)$ be the differential field with derivations $\Delta = \left\{\partial_x = \frac{\partial}{\partial x}, \partial_t = \frac{\partial}{\partial t}\right\}$. Consider the differential equation

$$\partial_x y = \frac{t}{x}y.$$

In the usual Picard–Vessiot theory, one forms the differential field generated by the entries of a fundamental solution matrix and all their derivatives (in fact, because the matrix satisfies the differential equation, we get the derivatives for free). We will proceed in a similar fashion here. The function

$$y = x^t$$

is a solution of the above equation. Although all derivatives with respect to x lie in the field $k(x^t)$, this is not true for $\partial_t(x^t) = (\log x)x^t$. Nonetheless, this is all that is missing and the derivations Δ naturally extend to the field

$$K = k(x^t, \log x),$$

the field gotten by adjoining to k a fundamental solution and its derivatives (of all orders) with respect to all the variables.

Let us now calculate the group $\mathrm{Gal}_\Delta(K/k)$ of k-automorphisms of K commuting with both ∂_x and ∂_t. Let $\sigma \in \mathrm{Gal}_\Delta(K/k)$. We first note that $\partial_x(\sigma(x^t)(x^t)^{-1}) = 0$ so $\sigma(x^t) = a_\sigma x^t$ for some $a_\sigma \in K$ with $\partial_x a_\sigma = 0$, *i.e.,* $a_\sigma \in C_K^{\{\partial_x\}} = C_k^{\{\partial_x\}} = \mathbb{C}(t)$. Next, a calculation shows that $\partial_x(\sigma(\log x) - \log x) = 0 = \partial_t(\sigma(\log x) - \log x)$ so we have that $\sigma(\log x) = \log x + c_\sigma$ for some $c_\sigma \in \mathbb{C}$. Finally, a calculation shows that

$$0 = \partial_t(\sigma(x^t)) - \sigma(\partial_t(x^t)) = (\partial_t a_\sigma - a_\sigma c_\sigma)x^t$$

so we have that

$$\partial_t\left(\frac{\partial_t a_\sigma}{a_\sigma}\right) = 0. \tag{3.1}$$

Conversely, one can show that for any a such that $\partial_x a = 0$ and equation (3.1) holds, the map defined by $x^t \mapsto ax^t$, $\log x \mapsto \log x + \frac{\partial_t a}{a}$ defines a differential k-automorphism of K so we have

$$\mathrm{Gal}_\Delta(K/k) = \left\{a \in C_K^{(\frac{\partial}{\partial x})} = C_k^{(\frac{\partial}{\partial x})} \mid a \neq 0 \text{ and } \partial_t\left(\tfrac{\partial_t a}{a}\right) = 0\right\}.$$

□

This example illustrates two facts. The first is that the Galois group of a parameterized linear differential equation is a group of $n \times n$ matrices (here $n = 1$) whose entries are functions of the parameters (in this case, t) satisfying certain differential equations; such a group is called a linear differential algebraic group (see Definition 3.3 below). In general, the Galois group of a parameterized linear differential equation will be such a group.

The second fact is that in this example $\mathrm{Gal}_\Delta(K/k)$ does not contain enough elements to give a Galois correspondence. Expressing an element of $\mathbb{C}(t)$ as $a = a_0\prod(t - b_i)^{n_i}$, $a_0, b_i \in \mathbb{C}, n_i \in \mathbb{Z}$, one can show that if $a \in \mathrm{Gal}_\Delta(K/k)$ then $a \in \mathbb{C}$, that is $\mathrm{Gal}_\Delta(K/k) = \mathbb{C}^*$. If $\sigma \in \mathrm{Gal}_\Delta(K/k)$ and $\sigma(x^t) = ax^t$ with $a \in \mathbb{C}$, then

$$\sigma(\log x) = \sigma\left(\frac{\partial_t x^t}{x^t}\right) = \frac{\partial_t(ax^t)}{ax^t} = \log x.$$

Therefore $\log x$ is fixed by the Galois group and so there cannot be a Galois correspondence. The problem is that we do not have an element $a \in \mathbb{C}(t)$ such that $\partial_t\left(\frac{\partial_t a}{a}\right) = 0$ and $\partial_t a \neq 0$.

In the Picard–Vessiot theory, one avoids a similar problem by insisting that the constant subfield is large enough, *i.e.,* algebraically closed. This insures that any consistent set of polynomial equations with constant coefficients will have a solution in the field. In the parameterized Picard–Vessiot theory that we will develop, we will need to insure that any consistent system of differential equations (with respect to the parametric variables) has a solution. This motivates the following definition.

Let k be a Δ-field with derivations $\Delta = \{\partial_1, \dots, \partial_m\}$. The *$\Delta$-ring $k\{y_1, \dots, y_n\}_\Delta$ of differential polynomials in n variables over k* is the usual polynomial ring in the infinite set of variables

$$\{\partial_1^{n_1}\partial_2^{n_2}\dots\partial_m^{n_m}y_j\}_{j=1,\dots,n}^{n_i\in\mathbb{N}}$$

with derivations extending those in Δ on k and defined by

$$\partial_i(\partial_1^{n_1}\dots\partial_i^{n_i}\dots\partial_m^{n_m}y_j)=\partial_1^{n_1}\dots\partial_i^{n_i+1}\dots\partial_m^{n_m}y_j.$$

Definition 3.2. We say that a Δ-field k is *differentially closed* if for any n and any set $\{P_1(y_1,\dots,y_n),\dots,P_r(y_1,\dots,y_n),Q(y_1,\dots,y_n)\}\subset k\{y_1,\dots,y_n\}_\Delta$, if the system

$$\{P_1(y_1,\dots,y_n)=0,\dots,P_r(y_1,\dots,y_n)=0,Q(y_1,\dots,y_n)\neq 0\}$$

has a solution in some Δ-field K containing k, then it has a solution in k

This notion was introduced by A. Robinson [41] and extensively developed by L. Blum [5] (in the ordinary differential case) and E. R. Kolchin [20] (who referred to these as constrainedly closed differential fields). More recent discussions can be found in [30] and [32]. A fundamental fact is that any Δ-field k is contained in a differentially closed differential field. In fact, for any such k there is a differentially closed Δ-field $\bar{k}$ containing k such that for any differentially closed Δ-field K containing k, there is a differential k-isomorphism of $\bar{k}$ into K. Differentially closed fields have many of the same properties with respect to differential fields as algebraically closed fields have with respect to fields but there are some striking differences. For example, the differential closure of a field has proper subfields that are again differentially closed. For more information, the reader is referred to the above papers.

Example 3.1 (bis). Let k be a $\Delta=\{\partial_x,\partial_t\}$-field and let $k_0=C_k^{\partial_x}$. Assume that k_0 is a differentially closed ∂_t-field and that $k=k_0(x)$ where $\partial_x x=1$ and $\partial_t x=0$. We again consider the differential equation

$$\partial_x y=\frac{t}{x}y$$

and let $K=k(x^t,\log x)$ where x^t, $\log x$ are considered formally as algebraically independent elements satisfying $\partial_t(x^t)=(\log x)x^t$, $\partial_x(x^t)=\frac{t}{x}x^t$, $\partial_t(\log x)=0$, $\partial_x(\log x)=\frac{1}{x}$. One can show that $C_K^{\{\partial_x\}}=k_0$ and that the Galois group is again

$$\mathrm{Gal}_\Delta(K/k)=\left\{a\in k_o^*\mid \partial_t\left(\tfrac{\partial_t(a)}{a}\right)=0\right\}.$$

Note that $\mathrm{Gal}_\Delta(K/k)$ contains an element a such that $\partial_t a\neq 0$ and $\partial_t\left(\frac{\partial_t(a)}{a}\right)=0$. To see this, note that the $\{\partial_t\}$-field $k_0(u)$, where u is transcendental over k_0 and $\partial_t u=u$ is a $\{\partial_t\}$-extension of k_0 containing such an element (*e.g.,* u). The definition of differentially closed ensures that k_0 also contains such an element. This implies that $\log x$ is not left fixed by $\mathrm{Gal}_\Delta(K/k)$. In fact, we will show in Section 4 that the following is a complete list of differential algebraic subgroups of $\mathrm{Gal}_\Delta(K/k)$ and the corresponding Δ-subfields of K:

Field	Group
$k((x^t)^n, \log x), n \in \mathbb{N}_{>0}$	$\{a \in k_0^* \mid a^n = 1\} = \mathbb{Z}/n\mathbb{Z}$
$k(\log x)$	$\{a \in k_0^* \mid \partial_t a = 0\}$
k	$\{a \in k_0^* \mid \partial_t(\partial_t(a)/a) = 0\}$

□

We now turn to stating the Fundamental Theorem in the Galois theory of parameterized linear differential equations. We need to give a formal definition of the kinds of groups that can occur and also of what takes the place of a Picard–Vessiot extension. This is done in the next two definitions.

Definition 3.3. Let k be a differentially closed Δ-differential field.

(1) A subset $X \subset k^n$ is said to be *Kolchin-closed* if there exists a set $\{f_1, \ldots, f_r\}$ of differential polynomials in n variables such that $X = \{a \in k^n \mid f_1(a) = \cdots = f_r(a) = 0\}$.

(2) A subgroup $G \subset \mathrm{GL}_n(k) \subset k^{n^2}$ is a *linear differential algebraic group* if $G = X \cap \mathrm{GL}_n(k)$ for some Kolchin-closed subset of k^{n^2}.

In the previous example, the Galois group was exhibited as a linear differential algebraic subgroup of $\mathrm{GL}_1(k_0)$. For any linear algebraic group G, the group $G(k)$ is a linear differential algebraic group. Furthermore, the group $G(C_k^\Delta)$ of constant points of G is also a linear differential algebraic group since it is defined by the (algebraic) equations defining G as well as the (differential) equations stating that the entries of the matrices are constants. We will give more examples in the next section

In the next definition, we will use the following conventions. If F is a $\Delta = \{\partial_0, \partial_1, \ldots, \partial_m\}$-field, we denote by C_F^0 the ∂_0 constants of F, that is, $C_F^0 = C_F^{\{\partial_0\}} = \{c \in F \mid \partial_0 c = 0\}$. One sees that C_F^0 is a $\Pi = \{\partial_1, \ldots, \partial_m\}$-field. We will use the notation $k\langle z_1, \ldots, z_r\rangle_\Delta$ to denote a Δ-field containing k and elements $z_1, \ldots, z_r$ such that no proper Δ-field has this property, *i.e.*,$k\langle z_1, \ldots, z_r\rangle_\Delta$ is the field generated over k by $z_1, \ldots, z_n$ and their higher derivatives.

Definition 3.4. Let k be a $\Delta = \{\partial_0, \partial_1, \ldots, \partial_m\}$-field and let

$$\partial_0 Y = AY$$

be a differential equation with $A \in \mathrm{gl}_n(k)$.

(1) A Δ-extension K of k is a *parameterized Picard–Vessiot extension of* k (or, more compactly, a PPV-extension of k) if $K = k\langle z_{1,1}, \ldots, z_{n,n}\rangle_\Delta$ where

  (a) the $z_{i,j}$ are entries of a matrix $Z \in \mathrm{GL}_n(K)$ satisfying $\partial_0 Z = AZ$, and

  (b) $C_K^0 = C_k^0$, *i.e.,* the ∂_0-constants of K coincide with the ∂_0-constants of k.

(2) The group $\mathrm{Gal}_\Delta(K/k) = \{\sigma : K \to K \mid \sigma$ is a k-automorphism such that $\sigma\partial = \partial\sigma$ for all $\partial \in \Delta\}$ is called the *parameterized Picard–Vessiot group* (PPV-group) associated with the PPV-extension K of k.

We note that if K is a PPV-extension of k and Z is as above then for any $\sigma \in \mathrm{Gal}_\Delta(K/k)$ one has that $\partial_0(\sigma(Z)Z^{-1}) = 0$. Therefore we can identify each $\sigma \in \mathrm{Gal}_\Delta(K/k)$ with a matrix in $GL_n(C_k^0)$. We can now state the Fundamental Theorem of parameterized Picard–Vessiot extensions

Theorem 3.5. *Let k be a $\Delta = \{\partial_0, \partial_1, \ldots, \partial_m\}$-field and assume that C_k^0 is a differentially closed $\Pi = \{\partial_1, \ldots, \partial_m\}$-field. Let*

$$\partial_0 Y = AY \tag{3.2}$$

be a differential equation with $A \in \mathrm{gl}_n(k)$.

(1) *There exists a PPV-extension K of k associated with* (3.2) *and this is unique up to differential k-isomorphism.*

(2) *The PPV-group* $\mathrm{Gal}_\Delta(K/k)$ *may be identified with $G(C_k^0)$, where G is a linear differential algebraic group defined over C_k^0.*

(3) *The map that sends any Δ-subfield F, $k \subset F \subset K$, to the group* $\mathrm{Gal}_\Delta(K/F)$ *is a bijection between differential subfields of K containing k and Kolchin-closed subgroups of* $\mathrm{Gal}_\Delta(K/k)$*. Its inverse is given by the map that sends a Kolchin-closed group H to the field $K^H = \{z \in K \mid \sigma(z) = z$ for all $\sigma \in H\}$.*

(4) *A Kolchin-closed subgroup H of* $\mathrm{Gal}_\Delta(K/k)$ *is a normal subgroup of* $\mathrm{Gal}_\Delta(K/k)$ *if and only if the field K^H is left set-wise invariant by* $\mathrm{Gal}_\Delta(K/k)$*. If this is the case, the map* $\mathrm{Gal}_\Delta(K/k) \to \mathrm{Gal}_\Delta(K^H/k)$ *is surjective with kernel H and K^H is a PPV-extension of k with PPV-group isomorphic to* $\mathrm{Gal}_\Delta(K/k)/H$*. Conversely, if F is a differential subfield of K containing k and F is a PPV-extension of k, then* $\mathrm{Gal}_\Delta(F/K)$ *is a normal subgroup of* $\mathrm{Gal}_\Delta(K/k)$*.*

The proof of this result is virtually the same as for the corresponding result of Picard–Vessiot theory. We give the details in Appendices 9.1–9.4.

We will give two simple applications of this theorem. For the first, let K be a PPV-extension of k corresponding to the equation $\partial_0 Y = AY$ and let $K = k\langle Z\rangle_\Delta$, where $Z \in \mathrm{GL}_n(K)$ and $\partial_0 Z = AZ$. We now consider the field $K_A^{\mathrm{PV}} = k(Z) \subset K$. Note that K_A^{PV} is not necessarily a Δ-field but it is a $\{\partial_0\}$-field. One can easily see that it is a PV-extension for the equation $\partial_0 Y = AY$ and that the PPV-group leaves it invariant and acts as $\{\partial_0\}$-automorphisms. We therefore have an injective homomorphism of $\mathrm{Gal}_\Delta(K/k) \to \mathrm{Gal}_{\{\partial_0\}}(K_A^{\mathrm{PV}}/k)$, defined by restriction. We then have the following result

Proposition 3.6. *Let k, C_k^0, K and K_A^{PV} be as above. Then:*

(1) *When considered as ordinary $\{\partial_0\}$-fields, K_A^{PV} is a PV-extension of k with algebraically closed field C_k^0 of ∂_0-constants.*

(2) *If* $\mathrm{Gal}_{\{\partial_0\}}(K_A^{\mathrm{PV}}/k) \subset \mathrm{GL}_n(C_k^0)$ *is the Galois group of the ordinary differential field* K_A^{PV} *over* k*, then the Zariski closure of the Galois group* $\mathrm{Gal}_\Delta(K/k)$ *in* $\mathrm{GL}_n(C_k^0)$ *equals* $\mathrm{Gal}_{\{\partial_0\}}(K_A^{\mathrm{PV}}/k)$.

Proof. Since a differentially closed field is algebraically closed, we have already justified the first statement. Clearly, $\mathrm{Gal}_\Delta(K/k) \subset \mathrm{Gal}_{\{\partial_0\}}(K/k)$. Since $\mathrm{Gal}_\Delta(K/k)$ and $\mathrm{Gal}_{\{\partial_0\}}(K_A^{\mathrm{PV}}/k)$ have the same fixed field k, the second statement follows. □

Remark 3.7. Fix a PPV-extension K of k and let $K = k\langle z_{1,1}, \dots, z_{n,n}\rangle_\Delta$ where the $z_{i,j}$ are entries of a matrix $Z \in \mathrm{GL}_n(K)$ satisfying $\partial_0 Z = AZ$ with $A \in \mathrm{gl}_n(k)$. One sees that the field K_A^{PV} defined above is independent of the particular invertible solution Z of $\partial_0 Y = AY$ used to generate K (although the Galois groups are only determined up to conjugacy). On the other hand, K_A^{PV} does depend on the equation $\partial_0 Y = AY$ and not just on the field K, that is if K is a PPV-extension of k for two different equations $\partial_0 Y = A_1 Y$ and $\partial_0 Y = A_2 Y$ with solutions Z_1 and Z_2 respectively, the fields $K_{A_1}^{\mathrm{PV}}$ and $K_{A_2}^{\mathrm{PV}}$ (and their respective PV-groups) may be very different. We will give an example of this in Remark 7.3.

Our second application is to characterize those equations $\partial_0 Y = AY$ whose PPV-groups are the set of Δ-constant points of a linear algebraic group. We first make the following definition.

Definition 3.8. Let k be a Δ-differential field and let $A \in \mathrm{gl}_n(k)$. We say that $\partial_0 Y = AY$ is *completely integrable* if there exist $A_i \in \mathrm{gl}_n(k)$, $i = 0, \dots, n$ with $A_0 = A$ such that

$$\partial_j A_i - \partial_i A_j = A_j A_i - A_i A_j \quad \text{for all } i, j = 0, \dots \text{ n.}$$

The nomenclature is motivated by the fact that the latter conditions on the A_i are the usual integrability conditions and the system of differential equations $\partial_i Y = A_i Y$, $i = 0, \dots, n$ are as in equations (2.1).

Proposition 3.9. *Let* k *be a* Δ*-differential field and assume that* k_0 *is a* Π*-differentially closed* Π*-field. Let* $A \in \mathrm{gl}_n(k)$ *and let* K *be a PPV-extension of* k *for* $\partial_0 Y = AY$*. Finally, let* $C = C_k^\Delta$.

(1) *There exists a linear algebraic group* G *defined over* C *such that* $\mathrm{Gal}_\Delta(K/k)$ *is conjugate to* $G(C)$ *if and only if* $\partial_0 Y = AY$ *is completely integrable. If this is the case, then* K *is a PV-extension of* k *corresponding to this integrable system.*

(2) *If* $A \in \mathrm{gl}_n(C_k^\Pi)$*, then* $\mathrm{Gal}_\Delta(K/k)$ *is conjugate to* $G(C)$ *for some linear algebraic group defined over* C.

Proof. (1) Let $K = k\langle Z\rangle_\Delta$ where $Z \in \mathrm{GL}_n(K)$ satisfies $\partial_0 Z = AZ$. If the PPV-group is as described, then there exists a $B \in \mathrm{GL}_n(C_k^0)$ such that $B\mathrm{Gal}_\Delta(K/k)B^{-1} = G(C)$, G an algebraic subgroup of $\mathrm{GL}_n(C_k^0)$, defined over C. Set $W = ZB^{-1}$. One sees that

$\partial_0 W = AW$ and $K = k\langle W\rangle_\Delta$. A simple calculation shows that for any $i = 0, \dots, n$, $\partial_i W \cdot W^{-1}$ is left fixed by all elements of the PPV-group. Therefore $\partial_i W = A_i W$ for some $A_i \in \mathrm{gl}_n(k)$. Since the ∂_i commute, one sees that the A_i satisfy the integrability conditions.

Now assume that there exist $A_i \in \mathrm{gl}_n(k)$ as above satisfying the integrability conditions. Let K be a PV-extension of k for the corresponding integrable system. From Lemma 9.9 in the Appendix, we know that K is also a PPV-extension of k for $\partial_0 Y = AY$. Let $\sigma \in \mathrm{Gal}_\Delta(K/k)$ and let $\sigma(Z) = ZD$ for some $D \in \mathrm{GL}_n(C_k^0)$. Since $\partial_i Z \cdot Z^{-1} = A_i \in \mathrm{GL}_n(k)$, we have that $\sigma(\partial_i Z \cdot Z^{-1}) = \partial_i Z \cdot Z^{-1}$. A calculation then shows that $\partial_i(D) = 0$. Therefore $D \in \mathrm{GL}_n(C_K^\Delta)$. We now need to show that $C_K^\Delta = C_k^\Delta$. This is clear since $C_K^\Delta \subset C_K^0 = C_k^0$. The final claim of Part (1) is now clear.

(2) Under the assumptions, the matrices $A_0 = A, A_1 = 0, \dots, A_n = 0$ satisfy the integrability conditions, so the conclusion follows from Part (1) above. □

If A has entries that are analytic functions of $x, t_1, \dots, t_m$, the fact that $\mathrm{Gal}_\Delta(K/k) = G(C)$ for some linear algebraic group does not imply that, for some open set of values $\vec{\tau} = (\tau_1, \dots, \tau_m)$ of $(t_1, \dots, t_m)$, the Galois group $G_{\vec{\tau}}$ of the ordinary differential equation $\partial_x Y = A(x, \tau_1, \dots, \tau_m)Y$ is independent of the choice of $\vec{\tau}$. We shall see in Section 5 that for equations with regular singular points we do have a constant Galois group (on some open set of parameters) if the PPV-group is $G(C)$ for some linear algebraic group but the following shows that this is not true in general.

Example 3.10. Let $\Pi = \{\partial_1 = \frac{\partial}{\partial t_1}, \partial_2 = \frac{\partial}{\partial t_2}\}$ and k_0 be a differentially closed Π-field containing $\mathbb{C}$. Let $k = k_0(x)$ be a $\Delta = \{\partial_0 = \frac{\partial}{\partial x}, \partial_1, \partial_2\}$-field where $\partial_0|_{k_0} = 0$, $\partial_0(x) = 1$, and ∂_1, ∂_2 extend the derivations on k_0 and satisfy $\partial_1(x) = \partial_2(x) = 0$. The equation

$$\frac{\partial Y}{\partial x} = A(x, t_1, t_2)Y = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} Y$$

has solution

$$Y = \begin{pmatrix} e^{t_1 x} & 0 \\ 0 & e^{t_2 x} \end{pmatrix}$$

One easily checks that

$$A_1 = \frac{\partial Y}{\partial t_1} Y^{-1} \in \mathrm{gl}_2(k) \quad \text{and} \quad A_2 = \frac{\partial Y}{\partial t_2} Y^{-1} \in \mathrm{gl}_2(k)$$

so the Galois group associated to this equation is conjugate to $G(C)$ for some linear algebraic group G (in fact $G(C) = C^* \times C^*$). Nonetheless, for fixed values $\vec{\tau} = (\tau_1, \tau_2) \in C^2$, the Galois group of $\partial_0 Y = A(x, \tau_1, \tau_2)Y$ is $G(C)$ if and only if τ_1 and τ_2 are linearly independent over the rational numbers. □

For more information on how a differential Galois group can vary in a family of linear differential equations see [1] §3.3, [2], [3], [4], [18], and [46].

We end this section with a result concerning solving parameterized linear differential equations in "finite terms". The statement of the result is the same *mutatis mutandi* as the corresponding result in the usual Picard–Vessiot theory (*cf.,* [40], Ch. 1.5) and will be proved in the Appendix.

Definition 3.11. Let k be a $\Delta = \{\partial_0, \dots \partial_m\}$-field. We say that a Δ-field L is a *parameterized liouvillian extension* of k if $C_L^0 = C_k^0$ and there exist a tower of Δ-fields $k = L_0 \subset L_1 \subset \cdots \subset L_r$ such that $L \subset L_r$ and $L_i = L_{i-1}\langle t_i \rangle_\Delta$ for $i = 1 \dots r$, where either

(1) $\partial_0 t_i \in L_{i-1}$, that is t_i is a *parameterized integral* (*of an element of* L_{i-1}), or

(2) $t_i \neq 0$ and $\partial_0 t_i / t_i \in L_{i-1}$, that is t_i is a *parameterized exponential* (*of an integral of an element in* L_{i-1}), or

(3) t_i is algebraic over L_{i-1}.

In Section 9.5 we shall prove a result (Lemma 9.14) that implies that a parmeterized liouvillian extension is an inductive limit of ∂_0-liouvillian extension (in the usual sense, *cf.,* Ch. 1.5, [40]). We will use this to prove the following result

Theorem 3.12. *Let k be a Δ-field and assume that C_k^0 is a differentially closed $\Pi = \{\partial_1, \dots \partial_m\}$-field. Let K be a PPV-extension of k with PPV-group G. The following are equivalent*

(1) *G contains a solvable subgroup* (*in the sense of abstract groups*) *H of finite index.*

(2) *K is a parameterized liouvillian extension of k.*

(3) *K is contained in a parameterized liouvillian extension of k.*

4 Linear differential algebraic groups

In this section we review some known facts concerning linear differential algebraic groups and give some examples of these groups. The theory of linear differential algebraic groups was initiated by P. Cassidy in [9] and further developed in [10]–[14]. The topic has also been addressed in [7], [22], [34], [36], [23], [47], and [48]. For a general overview see [8].

Let k_0 be a differentially closed $\Pi = \{\partial_1, \dots, \partial_m\}$-field and let $C = C_{k_0}^\Pi$. As we have already defined, a linear differential algebraic group is a Kolchin-closed subgroup of $\mathrm{GL}_n(k_0)$. Although the definition is a natural generalization of the definition of a linear algebraic group there are many points at which the theories diverge. The first is that an affine differential algebraic group (a Kolchin-closed subset X of k_0^m with group operations defined by everywhere defined rational differential functions) need not be a linear differential algebraic group although affine differential algebraic groups whose group laws are given by differential polynomial maps are linear differential algebraic groups [9]. Other distinguishing phenomena will emerge as we examine some examples.

Differential algebraic subgroups of $\boldsymbol{G_a^n}$. The group $\boldsymbol{G_a} = (k_0, +)$ is naturally isomorphic to $\left\{ \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) \mid a \in k_0 \right\}$ and, as such, has the structure of a linear differential algebraic group. Nonetheless we will continue to identify this group with k_0. The set $\boldsymbol{G_a^n} = (k_0^n, +)$ can also be seen to be a linear differential algebraic group. In ([9], Lemma 11), Cassidy shows that a subgroup H of $\boldsymbol{G_a^n}$ is a linear differential algebraic group if and only if H is the set of zeros of a set of linear homogeneous differential polynomials in $k_0\{y_1, \dots, y_n\}$. In particular, when $m = n = 1$, $\Pi = \{\partial\}$, the linear differential algebraic subgroups of $\boldsymbol{G_a}$ are all of the form

$$\boldsymbol{G_a^L}(k_0) = \{a \in \boldsymbol{G_a}(k_0) \mid L(a) = 0\}$$

where L is a linear differential operator (*i.e.,* an element of the ring $k_0[\partial]$ whose multiplication is given by $\partial \cdot a = a\partial + \partial(a)$). The lattice structure of these subgroups is given by

$$\boldsymbol{G_a^{L_1}}(k_0) \subset \boldsymbol{G_a^{L_2}}(k_0) \Leftrightarrow L_2 = L_3 L_1 \text{ for some } L_3 \in k_0[\partial].$$

Differential algebraic subgroups of $\boldsymbol{G_m^n}$. These have been classified by Cassidy ([9], Ch.IV).We shall restrict ourselves to the case $n = m = 1$, $\Pi = \{\partial\}$, that is, differential algebraic subgroups of $\boldsymbol{G_m}(k_0) = \mathrm{GL}_1(k_0) = k_0^*$. Any such group is either

(1) finite and cyclic, or

(2) $\boldsymbol{G_m^L} = \left\{a \in \boldsymbol{G_m}(k_0) \mid L\left(\frac{\partial a}{a}\right) = 0\right\} \text{ for some } L \in k_0[\partial]\right\}$.

For example, if $L = \partial$, the group

$$\boldsymbol{G_m^\partial}(k) = \left\{a \in k_0^* \mid \partial\left(\frac{\partial a}{a}\right) = 0\right\}$$

is the PPV-group of the parameterized linear differential equation $\partial_x y = \frac{t}{x} y$ where $\partial = \partial_t$. Notice that the only proper differential algebraic subgroup of $\{a \in k_0 \,|\partial a = 0\}$ is $\{0\}$. Therefore the only proper differential algebraic subgroups of $\boldsymbol{G_m^\partial}$ are either the finite cyclic groups, or $\boldsymbol{G_m}(C)$. This justifies the left column in the table given in Example 3.1 (bis). The right column follows by calculation.

The proof that the groups of (1) and (2) are the only possibilities proceeds in two steps. The first is to show that if the group is not connected (in the Kolchin topology where closed sets are Kolchin-closed sets), it must be finite (and therefore cyclic). The second step involves the *logarithmic derivative map* $l\partial \colon \boldsymbol{G_m}(k_0) \to \boldsymbol{G_a}(k_0)$ defined by

$$l\partial(a) = \frac{\partial a}{a}.$$

This map is a differential rational map (*i.e.,* the quotient of differential polynomials) and is a homomorphism. Furthermore, it can be shown that the following sequence is exact:

$$(1) \longrightarrow \boldsymbol{G_m}(C) \longrightarrow \boldsymbol{G_m}(k_0) \longrightarrow \boldsymbol{G_a}(k_0) \longrightarrow (0)$$
$$a \longmapsto \frac{\partial a}{a}.$$

The result then follows from the classification of differential subgroups of $\boldsymbol{G_a}(k_0)$. Note that in the usual theory of linear algebraic groups, there are no nontrivial rational homomorphisms from $\boldsymbol{G_m}$ to $\boldsymbol{G_a}$.

Semisimple differential algebraic groups. These groups have been classified by Cassidy in [14]. Buium [7] and Pillay [36] have given simplified proofs in the ordinary case (*i.e.,* $m = 1$). Buium's proof is geometric using the notion of jet groups and Pillay's proof is model theoretic and assumes from the start that the groups are finite dimensional (of finite Morely rank).

We say that a connected differential algebraic group is semisimple if it has no nontrivial normal Kolchin-connected, commutative subgroups. Let us start by considering semisimple differential algebraic subgroups G of $\mathrm{SL}_2(k_0)$. Let H be the Zariski-closure of such a group. If $H \neq \mathrm{SL}_2(k_0)$, then H is solvable (*cf.,* [40], p. 127) and so the same would be true of G. Therefore G must be Zariski-dense in $\mathrm{SL}_2(k_0)$. In [9] Proposition 42, Cassidy classified the Zariski-dense differential algebraic subgroups of $\mathrm{SL}_n(k_0)$. Let $\mathbb{D}$ be the k_0-vector space of derivations spanned by Π.

Proposition 4.1. *Let G be a proper Zariski-dense differential algebraic subgroup of* $\mathrm{SL}_n(k_0)$. *Then there exists a finite set $\Delta_1 \subset \mathbb{D}$ of commuting derivations such that G is conjugate to* $\mathrm{SL}_n(C_{k_0}^{\Delta_1})$, *the Δ_1-constant points of* $\mathrm{SL}_n(k_0)$.

Note that in the ordinary case $m = 1$, we can restate this more simply: *A proper Zariski-dense subgroup of* $\mathrm{SL}_n(k_0)$ *is conjugate to* $\mathrm{SL}_n(C)$. A complete classification of differential subgroups of SL_2 is given in [48]. The complete classification of semisimple differential algebraic groups is given by the following result (see [14], Theorem 18). By a Chevalley group, we mean a connected simple $\mathbb{Q}$-group containing a maximal torus diagonalizable over $\mathbb{Q}$.

Proposition 4.2. *Let G be a Kolchin-connected semisimple linear*[2] *differential algebraic group. Then there exist finite subsets of commuting derivations $\Delta_1, \ldots, \Delta_r$ of $\mathbb{D}$, Chevalley groups $H_1, \ldots, H_r$ and a differential isogeny* $\sigma\colon H_1(C_{k_0}^{\Delta_1}) \times \cdots \times H_r(C_{k_0}^{\Delta_r}) \to G$.

[2] One need not assume that G is linear since Pillay [34] showed that a semisimple differential algebraic group is differentially isomorphic to a linear differential algebraic group.

5 Isomonodromic families

In this section we shall describe how isomonodromic families of linear differential equations fit into this theory of parameterized linear differential equations. We begin[3] with some definitions and follow the exposition of Sibuya [45], Appendix 5. Let $\mathcal{D}$ be an open subset of the Riemann sphere (for simplicity, we assume that the point at infinity is not in $\mathcal{D}$) and let $\mathcal{D}(\vec{\tau}, \vec{r}\,) = \prod_{h=1}^{p} D(\tau_h, \rho_h)$ where $\vec{r} = (\rho_1, \ldots, \rho_p)$ is a p-tuple of positive numbers, $\vec{\tau} = (\tau_1, \ldots, \tau_p) \in \mathbb{C}^p$ and $D(\tau_h, \rho_h)$ is the open disk in $\mathbb{C}$ of radius ρ_h centered at the point τ_h. We denote by $\mathcal{O}(\mathcal{D} \times \mathcal{D}(\vec{\tau}, \vec{r}\,))$ the ring of functions $f(x, \vec{t}\,)$ holomorphic on $\mathcal{D} \times \mathcal{D}(\vec{\tau}, \vec{r}\,)$. Let $A(x, \vec{t}\,) \in \mathrm{gl}_n(\mathcal{O}(\mathcal{D} \times \mathcal{D}(\vec{\tau}, \vec{r}\,))$ and consider the differential equation

$$\frac{\partial W}{\partial x} = A(x, \vec{t}\,)W \tag{5.1}$$

Definition 5.1. A *system of fundamental solutions* of (5.1) is a collection of pairs $\{D(x_j, \vec{r}_j), W_j(x, \vec{t}\,)\}$ such that

(1) the disks $D(x_j, \vec{r}_j)$ cover $\mathcal{D}$ and

(2) for each $\vec{t} \in \mathcal{D}(\vec{\tau}, \vec{r}\,)$ the $W_j(x, \vec{t}\,) \in \mathrm{GL}_n(\mathcal{O}(D(x_j, \vec{r}_j) \times \mathcal{D}(\vec{\tau}, \vec{r}\,))$ are solutions of (5.1).

We define $C_{i,j}(\vec{t}\,) = W_i(x, \vec{t}\,)^{-1} W_j(x, \vec{t}\,)$ whenever $D(x_i, \vec{r}_i) \cap D(x_j, \vec{r}_j) \neq \emptyset$ and refer to these as the *connection matrices* of the system of fundamental solutions.

Definition 5.2. The differential equation (5.1) is *isomonodromic on* $\mathcal{D} \times \mathcal{D}(\vec{\tau}, \vec{r}\,)$ if there exists a system $\{D(x_j, \vec{r}_j), W_j(x, \vec{t}\,)\}$ of fundamental solutions such that the connection matrices $C_{i,j}(\vec{t}\,)$ are independent of t.

We note that for a differential equation that is isomonodromic in the above sense, the monodromy around any path is independent of $\vec{t}$ as well. To see this let γ be a path in $\mathcal{D}$ beginning and ending at x_0 and let $D(x_1, \vec{r}_1), \ldots, D(x_s, \vec{r}_s), D(x_1, \vec{r}_1)$ be a sequence of disks covering the path so that $D(x_i, \vec{r}_i) \cap D(x_{i+1}, \vec{r}_{i+1}) \neq \emptyset$ and $x_0 \in D(x_1, \vec{r}_1)$. If one continues $W_1(x_1, \vec{t}\,)$ around the path then the resulting matrix $\tilde{W} = W_1(x_1, \vec{t}\,)C_{1,s}C_{s,s-1} \ldots C_{2,1}$. By assumption, the monodromy matrix $C_{1,s}C_{s,s-1} \ldots C_{2,1}$ is independent of $\vec{t}$.

For equations with regular singular points, the monodromy group is Zariski dense in the PV-group. The above comments therefore imply that for an isomonodromic family, there is a nonempty open set of parameters such that for these values the monodromy (and therefore the PV-group) is constant as the parameters vary in this set. Conversely, fix $x_0 \in \mathcal{D}$ and fix a generating set S for $\Pi_1(\mathcal{D}, x_0)$. Assume that, for each value $\vec{t} \in \mathcal{D}(\vec{t}, \vec{r}\,)$, we can select a basis of the solution space of (5.1) such that the monodromy matrices corresponding to S with respect to this basis are independent

[3]The presentation clearly could be cast in the language of vector bundles (see [3], [4], [26], [27]) but the approach presented here is more in the spirit of the rest of the paper.

of $\vec{t}$. Bolibruch (Proposition 1, [6]) has shown that under these assumptions (5.1) is isomonodromic in the above sense[4].

With these definitions, Sibuya shows ([45], Theorem A.5.2.3)

Proposition 5.3. *The differential equation* (5.1) *is isomonodromic on* $\mathcal{D} \times \mathcal{D}(\vec{\tau}, \vec{r}\,)$ *if and only if there exist p matrices* $B_h(x, \vec{t}\,) \in \mathrm{gl}_n(\mathcal{O}(\mathcal{D} \times \mathcal{D}(\vec{\tau}, \vec{r}\,)))$, $h = 1, \ldots, p$ *such that the system*

$$\begin{aligned} \frac{\partial W}{\partial x} &= A(x, \vec{t}\,)W \\ \frac{\partial W}{\partial t_h} &= B_h(x, \vec{t}\,)W \quad (h = 1, \ldots, p) \end{aligned} \tag{5.2}$$

is completely integrable.

Some authors use the existence of matrices B_i as in Proposition 5.3 as the definition of isomonodromic (*cf.,* [26]). Sibuya goes on to note that if $A(x, \vec{t}\,)$ is rational in x and if the differential equation has only regular singular points, then the $B_h(x, \vec{t}\,)$ will be rational in x as well (without the assumption of regular singular points one cannot conclude that the B_i will be rational in x.) This observation leads to the next proposition.

For any open set $\mathcal{U} \subset \mathbb{C}^p$, let $\mathcal{M}(\mathcal{U})$ be the field of functions meromorphic on $\mathcal{U}$. Note that $\mathcal{M}(\mathcal{U})$ is a $\Pi = \left\{\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_p}\right\}$-field. If $\mathcal{U}' \subset \mathcal{U}$ then there is a natural injection of $\mathrm{res}_{\mathcal{U},\mathcal{U}'} \colon \mathcal{M}(\mathcal{U}) \to \mathcal{M}(\mathcal{U}')$. We shall need the following result of Seidenberg [43], [44]: *Let* $\mathcal{U}$ *be an open subset of* $\mathbb{C}^p$ *and let F be a* Π*-subfield of* $\mathcal{M}(\mathcal{U})$ *containing* $\mathbb{C}$. *If E is* Π*-field containing F and finitely generated* (*as a* Π*-field*) *over* $\mathbb{Q}$, *then there exists a nonempty open set* $\mathcal{U}' \subset \mathcal{U}$ *and an isomorphism* $\phi \colon E \to \mathcal{M}(\mathcal{U})$ *such that* $\phi|_F = \mathrm{res}_{\mathcal{U},\mathcal{U}'}$.

Let $A(x, \vec{t}\,)$ be as above, assume the entries of A are rational in x and let F be the Π-field generated by the coefficients of powers of x that appear in the entries of A. Let k_0 be the differential closure of F. We consider $k = k_0(x)$ to be a $\Delta = \left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_p}\right\}$-field in the obvious way. Given open subsets $U_1 \subset U_2$ of the Riemann sphere, we say that U_1 is a *punctured subset of* U_2 if there exist a finite number of disjoint closed disks $D_1, \ldots, D_r \subset U_2$ such that $U_1 = U_2 \backslash \left(\bigcup_{i=1}^r D_i\right)$.

Proposition 5.4. *Let* $A(x, \vec{t}\,)$, k_0 *and k be as above. Assume that the differential equation*

$$\frac{\partial W}{\partial x} = A(x, \vec{t}\,)W \tag{5.3}$$

has only regular singular points. Then this equation is isomonodromic on $\mathcal{D}' \times \mathcal{U}$, *for some nonempty, open* $\mathcal{U} \subset \mathcal{D}(\vec{\tau}, \vec{r}\,)$ *and* $\mathcal{D}'$ *a punctured subset of D if and only if*

[4] Throughout [6], Bolibruch assumes that $A(x, \vec{t}\,) = \sum_{i=1}^s \frac{A_i(\vec{t}\,)}{x - t_i}$ but his proof of this result works *mutatis mutandi* for any equation with regular singular points.

the PPV-group of this equation over k is conjugate to $G(\mathbb{C})$ for some linear algebraic group G defined over $\mathbb{C}$. In this case, the monodromy group of (5.3) *is independent of $\vec{t} \in U$.*

Proof. Assume that (5.3) is isomonodromic. Proposition 5.3 and the comments after it ensure that we can complete (5.3) to a completely integrable system (5.2) where the $B_i(x, \vec{t}\,)$ are rational in x. The fact that this is a completely integrable system is equivalent to the coefficients of the powers of x appearing in the entries of the B_i satisfying a system $\mathcal{S}$ of Π-differential equations with coefficients in k_0. Since this system has a solution and k_0 is differentially closed, the system must have a solution in k_0. Therefore we may assume that the $B_i \in \mathrm{gl}_n(k)$. An application of Proposition 3.9 (1) yields the conclusion.

Now assume that the PPV-group is conjugate to $G(\mathbb{C})$ for some linear algebraic group G. Proposition 3.9 (1) implies that we can complete (5.3) to a completely integrable system (5.2) where the $B_i(x, \vec{t}\,)$ are in $\mathrm{gl}_n(k)$. Let E be the Π-field generated by the coefficients of powers of x appearing in the entries of A and the B_i. By the result of Seidenberg referred to above, there is a nonempty open set $\mathcal{U} \subset \mathcal{D}(\vec{\tau}, \vec{r}\,)$ such that these coefficients can be assumed to be analytic on $\mathcal{U}$.The matrices B_i have entries that are rational in x and so may have poles (depending on $\vec{t}$) in D. By shrinking $\mathcal{U}$ if necessary and replacing D with a punctured subset D' of D, we can assume that A and the B_i have entries that are holomorphic in $D' \times \mathcal{U}$. We now apply Proposition 5.3 to reach the conclusion. □

6 Second order systems

In this section we will apply the results of the previous four sections to give a classification of parameterized second order systems of linear differential equations. We will first consider the case of second order parameterized linear equations depending on only one parameter.

Proposition 6.1. *Let k be a $\Delta = \{\partial_0, \partial_1\}$-field, assume that $k_0 = C_k^0$ is a differentially closed $\Pi = \{\partial_1\}$-field and let $C = C_k^\Delta$. Let $A \in \mathrm{sl}_2(k)$ and let K be the PPV-extension corresponding to the differential equation*

$$\partial_0 Y = AY. \tag{6.1}$$

Then, either

(1) $\mathrm{Gal}_\Delta(K/k)$ *equals* $\mathrm{SL}_2(k_0)$*, or*

(2) $\mathrm{Gal}_\Delta(K/k)$ *contains a solvable subgroup of finite index and K is a parameterized liouvillian extension of k, or*

(3) $\mathrm{Gal}_\Delta(K/k)$ *is conjugate to* $\mathrm{SL}_2(C)$ *and there exist* $B_1 \in \mathrm{sl}_2(k)$ *such that the system*

$$\partial_0 Y = AY$$
$$\partial_1 Y = B_1 Y$$

is an integrable system.

Proof. Let $Z \in \mathrm{GL}_2(k)$ be a fundamental solution matrix of (6.1) and let $z = \det Z$. We have that $\partial_0 z = (\mathrm{trace} A)z$ ([40], Exercise 1.14.5), so $z \in k_0$. For any $\sigma \in \mathrm{Gal}_\Delta(K/k)$, $z = \sigma(z) = \det\sigma \cdot z$ so $\det\sigma = 1$. Therefore, $\mathrm{Gal}_\Delta \subset \mathrm{SL}_2(k_0)$. Let G be the Zariski-closure of $\mathrm{Gal}_\Delta(K/k)$. If $G \neq \mathrm{SL}_2(k_0)$, then G has a solvable subgroup of finite index and so the same holds for $\mathrm{Gal}_\Delta(K/k)$. Therefore, (2) holds. If $G = \mathrm{Gal}_\Delta(K/k) = \mathrm{SL}_2(k_0)$, then (1) holds. If $G = \mathrm{SL}_2(k_0)$ and $G \neq \mathrm{Gal}_\Delta(K/k)$, then Proposition 4.1 and the discussion immediately following it imply that there is a $B \in \mathrm{SL}_2(k_0)$ such that $B\mathrm{Gal}_\Delta(K/k)B^{-1} = \mathrm{SL}_2(C)$. Proposition 3.9 then implies that the parameterized equation $\partial_0 Y = AY$ is completely integrable, yielding conclusion (3). □

If the entries of A are functions of x and t, analytic in some domain and rational in x, we can combine the above proposition with Proposition 5.4 to yield the next corollary. Let $\mathcal{D}$ be an open region on the Riemann sphere and $D(\tau_0, \rho_0)$ be the open disk of radius ρ_0 centered at τ_0 in $\mathbb{C}$. Let $\mathcal{O}(\mathcal{D} \times D(\tau_0, \rho_0))$ be the ring of functions holomorphic in $\mathcal{D} \times D(\tau_0, \rho_0)$ and let $A(t, x) \in \mathrm{sl}_2(\mathcal{O}(\mathcal{D} \times D(\tau_0, \rho_0))$ and assume that $A(x, t)$ is rational in x. Let $\Delta = \left\{\partial_0 = \frac{\partial}{\partial x}, \partial_1 = \frac{\partial}{\partial t}\right\}$ and $\Pi = \{\partial_1\}$. Let k_0 be a differentially closed Π-field containing the coefficients of powers of x appearing in the entries of A and let $k = k_0(x)$ be the Δ-field gotten by extending ∂_1 via $\partial_1(x) = 0$ and defining ∂_0 to be zero on k_0 and $\partial_1(x) = 1$.

Corollary 6.2. *Let* $k_0, k, A(t, x)$ *be as above and let* K *be the PV-extension associated with*

$$\frac{\partial Y}{\partial x} = A(x, t)Y. \tag{6.2}$$

Then, either

(1) $\mathrm{Gal}_\Delta(K/k) = \mathrm{SL}_2(k_0)$*, or*

(2) $\mathrm{Gal}_\Delta(K/k)$ *contains a solvable subgroup of finite index and* K *is a parameterized liouvillian extension of* k*, or*

(3) *equation* (6.2) *is isomonodromic on* $D' \times \mathcal{U}$ *where* D' *is a punctured subset of* D *and* U *is an open subset of* $D(\tau_0, \rho_0)$.

We can also state a result similar to Proposition 6.1 for parameterized linear equations having more than one parameter. We recall that if k_0 is a $\Pi = \{\partial_1, \dots \partial_m\}$-field, we denote by $\mathbb{D}$ the k_0-vector space of derivations spanned by Π.

Proposition 6.3. *Let k be a $\Delta = \{\partial_0, \ldots \partial_m\}$-field, assume that $k_0 = C_k^0$ is a differentially closed $\Pi = \{\partial_1, \ldots \partial_m\}$-field. Let $A \in \mathrm{sl}_2(k)$ and let K be the PPV-extension corresponding to the differential equation*

$$\partial_0 Y = AY.$$

Then, either

(1) $\mathrm{Gal}_\Delta(K/k) = \mathrm{SL}_2(k_0)$, *or*

(2) $\mathrm{Gal}_\Delta(K/k)$ *contains a solvable subgroup of finite index and K is a parameterized liouvillian extension of k, or*

(3) $\mathrm{Gal}_\Delta(K/k)$ *is a proper Zariski-dense subgroup of* $\mathrm{SL}_2(k_0)$ *and there exist*

(a) *a commuting k_0-basis $\{\partial'_1, \ldots, \partial'_m\}$ of $\mathbb{D}$, and*

(b) *an integer r, $1 \le r \le m$ and elements $B_i \in \mathrm{gl}_2(k)$, $i = 1, \ldots r$,*

such that the system

$$\begin{aligned} \partial_0 Y &= AY \\ \partial'_1 Y &= B_1 Y \\ &\vdots \\ \partial'_r Y &= B_r Y \end{aligned}$$

is an integrable system.

Proof. The proof begins in the same way as that for Proposition 6.1 and Cases (1) and (2) are the same. If neither of these hold, then $\mathrm{Gal}_\Delta(K/k)$ is a proper Zariski-dense subgroup of $\mathrm{SL}_2(k_0)$ and so by Proposition 4.1, there exist commuting derivations $\Delta' = \{\partial'_1, \ldots, \partial'_r\} \subset \mathbb{D}$ such that $\mathrm{Gal}_\Delta(K/k)$ is conjugate to $\mathrm{SL}_2(C_k^\Pi)$. We may assume that the ∂'_i are k_0 independent. Proposition 7 of Chapter 0 of [22] implies that we can extend Δ' to a commuting basis of $\mathbb{D}$. After conjugation by an element $B \in \mathrm{GL}_2(k)$, we can assume that the PPV-group is $\mathrm{SL}_2(C_k^{\Delta'})$. A calculation shows that $(\partial'_i Y)Y^{-1}$ is left invariant by this group for $i = 1, \ldots, r$ and the conclusion follows. □

The third case of the previous proposition can be stated informally as: *After a change of variables in the parameter space, the parameterized differential equation is completely integrable with respect to x and a subset of the new parameters.*

7 Inverse problem

The general inverse problem can be stated as: *Given a differential field, which groups can occur as Galois groups of PPV-extensions of this field?* We have no definitive results but will give two examples in this section.

Example 7.1. Let k be a $\Delta = \{\partial_0, \partial_1\}$-field with $k_0 = C_k^{\{\partial_0\}}$ differentially closed and $k = k_0(x)$, $\partial_0(x) = 1$, $\partial_1(x) = 0$. We wish to know: *Which subgroups G of $\boldsymbol{G_a}(k_0)$ are Galois groups of PPV-extensions of k?* The answer is that all proper differential algebraic subgroups of $\boldsymbol{G_a}(k_0)$ appear in this way but $\boldsymbol{G_a}(k_0)$ itself cannot be the Galois group of a PPV-extension K of k.

We begin by showing that $\boldsymbol{G_a}(k_0)$ cannot be the Galois group of a PPV-extension K of k. In Section 9.4, we show that K is the differential function field of a G-torsor. If $G = \boldsymbol{G_a}(k_0)$, then the Corollary to Theorem 4 of Chapter VII.3 of [22] implies that this torsor is trivial and so $K = k\langle z\rangle_\Delta$ where $\sigma(z) = z + c_\sigma$ for all $\sigma \in \boldsymbol{G_a}(k_0)$. This further implies that $\partial_0(z) = a$ for some $a \in k$. Since $k = k_0(x)$ and k_0 is algebraically closed, we may write

$$a = P(x) + \sum_{i=1}^{r}\left(\sum_{j=1}^{s_i} \frac{b_{i,j}}{(x-c_i)^j}\right)$$

where $P(x)$ is a polynomial with coefficients in k_0 and the $b_{i,j}, c_i \in k_0$. Furthermore, there exists an element $R(x) \in k$ such that

$$\partial_0(z - R(x)) = \sum_{i=1}^{r} \frac{b_{i,1}}{(x-c_i)}$$

so after such a change, we may assume that

$$a = \sum_{i=1}^{r} \frac{b_i}{x-c_i}$$

for some $b_i, c_i \in k_0$.

We shall show that the Galois group of K over k is

$$\{z \in k_0 \mid L(z) = 0\}$$

where L is the linear differential equation in $k[\partial_1]$ whose solution space is spanned (over C) by $\{b_1, b_2, \dots, b_r\}$. In particular, *the group $\boldsymbol{G_a}(k_0)$ is not a Galois group of a PPV-extension of k.*

To do this form a new PPV-extension $F = k\langle z_1, \dots, z_r\rangle_\Delta$ where $\partial_0 z_i = \frac{1}{x-c_i}$. Clearly, there exists an element $w = \sum_{i=1}^r b_i z_i \in F$ such that $\partial_0 w = a$. Therefore we can consider K as a subfield of F. A calculation shows that $\partial_0\big(\partial_1 z_i + \frac{\partial_1 c_i}{x-c_i}\big) = 0$ so $\partial_1 z_i \in k$. Therefore Proposition 3.9 implies that the PPV-group $\mathrm{Gal}_\Delta(F/k)$ is of the form $G(C)$ for some linear algebraic group G and that F is a PV-extension of k. The Kolchin–Ostrowski Theorem ([21], p. 407) implies that the elements z_i are algebraically independent over k. The PPV-group $\mathrm{Gal}_\Delta(F/k)$ is clearly a subgroup of $\boldsymbol{G_a}(C)^r$ and since the transcendence degree of F over k must equal the dimension of this group, we have $\mathrm{Gal}_\Delta(F/k) = \boldsymbol{G_a}(C)^r$.

For $\sigma = (d_1, \dots, d_r) \in \boldsymbol{G_a}(C)^r = \mathrm{Gal}_\Delta(F/k)$, $\sigma(w) = w + \sum_{i=1}^r d_i b_i$. The Galois theory implies that restricting elements of $\mathrm{Gal}_\Delta(F/k)$ to K yields a surjective homomorphism onto $\mathrm{Gal}_\Delta(K/k)$, so we can identify $\mathrm{Gal}_\Delta(K/k)$ with the C-span of the b_i. Therefore $\mathrm{Gal}_\Delta(K/k)$ has the desired form.

We now show that any proper differential algebraic subgroup H of $\boldsymbol{G_a}(k_0)$ is the PPV-group of a PPV-extension of k. As stated in Section 4. $H = \{a \in \boldsymbol{G_a}(k_0) \mid L(a) = 0\}$ for some linear differential operator L with coefficients in k_0. Since k_0 is differentially closed, there exist $b_1, \ldots, b_m \in k_0$ linearly independent over $C = C_k^\Delta$ that span the solution space of $L(Y) = 0$. Let

$$a = \sum_{i=1}^{m} \frac{b_i}{x - i}.$$

The calculation above shows that the PPV-group of the PPV-extension of k for $\partial_0 y = a$ is H. □

The previous example leads to the question: Find a Δ-field k such that $\boldsymbol{G_a}(k_0)$ is a Galois group of a PPV-extension of k. We do this in the next example.

Example 7.2. Let k be a $\Delta = \{\partial_0, \partial_1\}$-field with $k_0 = C_k^{\{\partial_0\}}$ differentially closed and $k = k_0(x, \log x, x^{t-1}e^{-x})$, $\partial_0(x) = 1$, $\partial_0(\log x) = \frac{1}{x}$, $\partial_0(x^{t-1}e^{-x}) = \frac{t-x-1}{x} x^{t-1}e^{-x}$, $\partial_1(x) = \partial_1(\log x) = 0$, $\partial_1(x^{t-1}e^{-x}) = (\log x)x^{t-1}e^{-x}$. Consider the differential equation

$$\partial_0 y = x^{t-1}e^{-x}$$

and let K be the PPV-extension of k for this equation. We may write $K = k\langle\gamma\rangle_\Delta$, where γ satisfies the above equation (γ is known as the *incomplete Gamma function*). We have that $K = k(\gamma, \partial_1\gamma, \partial_1^2\gamma, \ldots)$. In [19], the authors show that $\gamma, \partial_1\gamma, \partial_1^2\gamma, \ldots$ are algebraically independent over k. Therefore, for any $c \in \boldsymbol{G_a}(k_0)$, $\partial_1^i\gamma \mapsto \partial_1^i\gamma + \partial_1^i c$, $i = 0, 1, \ldots$ defines an element of $\mathrm{Gal}_\Delta(K/k)$. Therefore $\mathrm{Gal}_\Delta(K/k) = \boldsymbol{G_a}(k_0)$.

Over $k_0(x)$, γ satisfies

$$\frac{\partial^2\gamma}{\partial x^2} - \frac{t-1-x}{x}\frac{\partial\gamma}{\partial x} = 0$$

and one can furthermore show that the PPV-group over $k_0(x)$ of this latter equation is

$$\begin{aligned} H &= \left\{ \left(\begin{smallmatrix} 1 & a \\ 0 & b \end{smallmatrix}\right) \mid a \in k_0,\ b \in k_0^*,\ \partial_1\left(\tfrac{\partial_1 b}{b}\right) = 0 \right\} \\ &= \boldsymbol{G_a}(k_0) \rtimes \boldsymbol{G_m}^{\partial_1}, \end{aligned}$$

where $\boldsymbol{G_m}^{\partial_1} = \left\{ b \in k_0^* \mid \partial_1\left(\frac{\partial_1 b}{b}\right) = 0 \right\}$. □

Remark 7.3. We can use the previous example to exhibit two equations $\partial_0 Y = A_1 Y$ and $\partial_0 Y = A_2 Y$ having the same PPV-extension K of k but such that $K_{A_1}^{\mathrm{PV}} \neq K_{A_2}^{\mathrm{PV}}$ and that these latter PV-extensions have different PV-groups (*cf.*, Remark 3.7). Let k

and γ be as in the above example and let

$$A_1 = \begin{pmatrix} 0 & x^{t-1}e^{-x} \\ 0 & 0 \end{pmatrix} \qquad A_2 = \begin{pmatrix} 0 & x^{t-1}e^{-x} & (\log x)x^{t-1}e^{-x} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We have that

$$Z_1 = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix} \qquad Z_2 = \begin{pmatrix} 1 & \gamma & \partial_1(\gamma) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

satisfy $\partial_0 Z_1 = A_1 Z_1$ and $\partial_0 Z_2 = A_2 Z_2$. K is the PPV-extension associated with either equation and the Galois group $\mathrm{Gal}_\Delta(K/k)$ is $\boldsymbol{G_a}(k_0)$. We have that $K^{\mathrm{PV}}_{A_1} = k(\gamma) \neq K^{\mathrm{PV}}_{A_2} = k(\gamma, \partial_1\gamma)$ since γ and $\partial_1\gamma$ are algebraically independent over k. With respect to the first equation, $\mathrm{Gal}_\Delta(K/k)$ is represented in $\mathrm{GL}_2(k_0)$ as

$$\left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \;\middle|\; c \in k_0 \right\}$$

and with respect to the second equation $\mathrm{Gal}_\Delta(K/k)$ is represented in $\mathrm{GL}_3(k_0)$ as

$$\left\{ \begin{pmatrix} 1 & c & \partial_1(c) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; c \in k_0 \right\}.$$

The image of $\boldsymbol{G_a}(k_0)$ in $\mathrm{GL}_2(k_0)$ is Zariski-closed while the Zariski closure of the image of $\boldsymbol{G_a}(k_0)$ in $\mathrm{GL}_3(k_0)$ is

$$\left\{ \begin{pmatrix} 1 & c & d \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; c, d \in k_0 \right\}.$$

As algebraic groups, the first group is just $\boldsymbol{G_a}(k_0)$ and the second is $\boldsymbol{G_a}(k_0) \times \boldsymbol{G_a}(k_0)$.

8 Final comments

Other Galois theories. In [37], Pillay proposes a Galois theory that extends Kolchin's Galois theory of strongly normal extensions. We will explain the connections to our results.

Let k be a differential field and K a Picard–Vessiot extension of k. K has the following property: for any differential extension E of K and any differential k-isomorphism ϕ of K into E, we have that $\phi(K) \cdot C = K \cdot C$, where C is the field of constants of E. Kolchin has shown ([21], Chapter VI) this is the key property for developing a Galois theory. In particular, he defines a finitely generated differential field extension K of k to be *strongly normal* if for any differential extension E of K

and any differential k-isomorphism of K into E we have that

(1) $\phi(K)\langle C\rangle = K\langle C\rangle$, where C are the constants of E and

(2) ϕ leaves each of the constants of K fixed.

For such fields, Kolchin shows that the differential Galois group of K over k has the structure of an algebraic group and that the usual Galois correspondence holds.

In [31], [35], [37], [38] Pillay considers *ordinary* differential fields and generalizes this theory. The key observation is that the condition (1) can be restated as

$$(1') \qquad \phi(K)\langle X(E)\rangle = K\langle X(E)\rangle,$$

where X is the differential algebraic variety defined by the equation $\partial Y = 0$ and $X(E)$ are the E-points of X. For X, any differential algebraic variety defined over k (or more generally, any Kolchin-constructible set), Pillay defines a differential extension K to be an X-*strongly normal* extension of k if for any differential extension E of K and any differential k-isomorphism of K into E we have that equation (1′) holds and that (2) is replaced by technical (but important) other conditions. Pillay then uses model theoretic tools to show that for these extensions, the Galois group is a finite dimensional differential algebraic group (note that in the PPV-theory, infinite dimensional differential algebraic groups can occur, *e.g.*, $\boldsymbol{G_a}$). The finite dimensionality results from the fact that the underlying differential fields are ordinary differential fields and that finite sets of elements in the differential closure of an ordinary differential field generate fields of finite transcendence degree (a fact that is no longer true for partial differential fields). Because of this, Pillay was able to recast his theory in [38] in the language of subvarieties of certain jet spaces. If one generalizes Pillay's definition of strongly normal to allow *partial* differential fields with derivations Δ and takes for X the differential algebraic variety defined by $\{\partial_Y = 0 \mid \partial \in \Pi\}$ where $\Pi \subset \Delta$, then this definition would include PPV-extensions. Presumably the techniques of [37] can be used to prove many of these results as well. Nonetheless, we feel that a description of the complete situation for PPV-fields is sufficiently self contained as to warrant an independent exposition.

Landesman [24] has been generalizing Kolchin's Galois theory of strongly normal extensions to differential fields having a designated subset of derivations acting as parametric derivations. When this is complete, many of our results should follow as a special case of his results.

Umemura [49]–[54] has proposed a Galois theory for general nonlinear differential equations. Instead of Galois groups, he uses Lie algebras to measure the symmetries of differential fields. Malgrange [28], [29] has proposed a Galois theory of differential equations where the role of the Galois group is taken by certain groupoids. Both Umemura and Malgrange have indicated to us that their theories can analyze parameterized differential equations as well.

Future directions. There are many questions suggested by the results presented here and we indicate a few of these.

(1) Deligne [15], [16] (see also [40]) has shown that the usual Picard–Vessiot theory can be presented in the language of Tannakian categories. Can one characterize in a similar way the category of representations of linear differential algebraic groups and use this to develop the parameterized Picard–Vessiot theory?[5]

(2) How does the parameterized monodromy sit inside the parameterized Picard–Vessiot groups? To what extent can one extend Ramis' characterization of the local Galois groups to the parameterized case?

(3) Can one develop algorithms to determine the Galois groups of parameterized linear differential equations? Sit [48] has classified the differential algebraic subgroups of SL_2. Can this classification be used to calculate Galois groups of second order parameterized differential equations in analogy to Kovacic's algorithm for second order linear differential equations?

(4) Characterize those linear differential algebraic groups that appear as Galois groups of $k_0(x)$ where k_0 is as in Example 7.1.

9 Appendix

In this Appendix, we present proofs of results that imply Theorem 3.5 and Theorem 3.12. In Section 3, Theorem 3.5 is stated for a parameterized system of *ordinary* linear differential equations but it is no harder to prove an analogous result for parameterized integrable systems of linear partial differential equations and we do this in this appendix. The first section contains a discussion of *constrained extensions*, a concept needed in the proof of the existence of PPV-extensions. In the next three sections, we prove results that simultaneously imply Theorem 2.1 and Theorem 3.5. The proofs are almost, word-for-word, the same as the proofs of the corresponding result for PV-extensions ([39], Ch. 1) once one has taken into account the need for subfields of constants to be differentially closed. Nonetheless we include the proofs for the convenience of the reader. The final section contains a proof of Theorem 3.12.

9.1 Constrained extensions

Before turning to the proof of Theorem 3.5, we shall need some more facts concerning differentially closed fields (see Definition 3.2). If $k \subset K$ are Δ-fields and $\eta = (\eta_1, \dots, \eta_r) \in K^r$, we denote by $k\{\eta\}_\Delta$ (resp. $k\langle\eta\rangle_\Delta$) the Δ-ring (resp. Δ-field) generated by k and $\eta_1, \dots, \eta_r$, that is, the ring (resp. field) generated by k and all the derivatives of the η_i. We shall denote by $k\{y_1, \dots, y_n\}_\Delta$ the ring of differential polynomials in n variables over k (*cf.,* Section 3). A k-Δ-isomorphism of $k\{\eta\}_\Delta$ is a k-isomorphism σ such that $\sigma\partial = \partial\sigma$ for all $\partial \in \Delta$.

[5] Added in proof: Alexey Ovchinnikov has done this and the details will appear in his forthcoming Ph.D. thesis at NC State University.

Definition 9.1. ([21], Ch. III.10; [20]) Let $k \subset K$ be Δ-fields.

(1) We say that a finite family of elements $\eta = (\eta_1, \dots, \eta_r) \subset K^r$ is *constrained* over k if there exist differential polynomials $P_1, \dots, P_s, Q \in k\{y_1, \dots, y_r\}_\Delta$ such that

  (a) $P_1(\eta_1, \dots, \eta_r) = \cdots = P_s(\eta_1, \dots, \eta_r) = 0$ and $Q(\eta_1, \dots, \eta_r) \neq 0$, and

  (b) for any Δ-field E, $k \subset E$, if $(\zeta_1, \dots, \zeta_r) \in E^r$ and $P_1(\zeta_1, \dots, \zeta_r) = \cdots = P_1(\zeta_1, \dots, \zeta_r) = 0$ and $Q(\zeta_1, \dots, \zeta_r) \neq 0$, then the map $\eta_i \mapsto \zeta_i$ induces a k-Δ-isomorphism of $k\{\eta_1, \dots, \eta_r\}_\Delta$ with $k\{\zeta_1, \dots, \zeta_r\}_\Delta$.

  We say that Q is the *constraint* of η over k.

(2) We say K is a *constrained* extension of k if every finite family of elements of K is constrained over k.

(3) We say k is *constrainedly closed* if k has no proper constrained extensions.

The following Proposition contains the facts that we will use:

Proposition 9.2. *Let $k \subset K$ be Δ-fields and $\eta \in K^r$*

(1) *η is* constrained *over k with constraint Q if and only if $k\{\eta, 1/Q(\eta)\}_\Delta$ is a simple Δ-ring,* i.e. *a Δ-ring with no proper nontrivial Δ-ideals.*

(2) *If η is constrained over k and $K = k\langle\eta\rangle_\Delta$, then any finite set of elements of K is constrained over k, that is, K is a constrained extension of k.*

(3) *K is differentially closed if and only if it is constrainedly closed.*

(4) *Every differential field has a constrainedly closed extension.*

One can find the proofs of these in [20], where Kolchin uses the term constrainedly closed instead of differentially closed. Proofs also can be found in [32] where the author uses a model theoretic approach. Item (1) follows from the fact that any maximal Δ-ideal in a ring containing $\mathbb{Q}$ is prime ([21], Ch. I.2, Exercise 3 or [40], Lemma 1.17.1) and that for any radical differential ideal I in $k\{y_1, \dots, y_r\}_\Delta$ there exist differential polynomials $P_1, \dots, P_s$ such that I is the smallest radical differential ideal containing $P_1, \dots, P_s$ (the Ritt–Raudenbusch Theorem [21], Ch. III.4). Item (2) is fairly deep and is essentially equivalent to the fact that the projection of a Kolchin-constructible set (an element in the boolean algebra generated by Kolchin-closed sets) is Kolchin-constructible. Items (3) and (4) require some effort but are not too difficult to prove. Generalizations to fields with noncommuting derivations can be found in [56] and [33].

In the usual Picard–Vessiot theory, one needs the following key fact: Let k be a differential field with algebraically closed subfield of constants C. If R is a simple differential ring, finitely generated over k, then any constant of R is in C (Lemma 1.17, [40]). The following result generalizes this fact and plays a similar role in the parameterized Picard–Vessiot theory. Recall that if k is a $\Delta = \{\partial_0, \dots, \partial_m\}$-field and $\Lambda \subset \Delta$, we denote by C_k^Λ the set $\{c \in k \mid \partial c = 0 \text{ for all } \partial \in \Lambda\}$. One sees that C_k^Λ is a $\Pi = \Delta \backslash \Lambda$-field.

Lemma 9.3. *Let $k \subset K$ be Δ-fields, $\Lambda \subset \Delta$, and $\Pi = \Delta \backslash \Lambda$. Assume that C_k^Λ is Π-differentially closed. If K is a Δ-constrained extension of k, then $C_K^\Lambda = C_k^\Lambda$.*

Proof. Let $\eta \in C_K^\Lambda$. Since K is a Δ-constrained extension of k, there exist $P_1, \dots, P_s$, $Q \in k\{y\}_\Delta$ satisfying the conditions of Definition 9.1 with respect to η and k. We will first show that there exist $P_1, \dots, P_s, Q \in C_k^\Lambda\{y\}_\Delta$ satisfying the conditions of Definition 9.1 with respect to η and k.

Let $\{\beta_i\}_{i \in I}$ be a C_k^Λ-basis of k. Let $R \in k\{y\}_\Delta$ and write $R = \sum R_i \beta_i$ where each $R_i \in C_k^\Lambda\{y\}_\Delta$. Since linear independence over constants is preserved when one goes to extension fields ([21], Ch. II.1), for any differential Δ-extension E of k and $\zeta \in C_E^\Lambda$, we have that $R(\zeta) = 0$ if and only if all $R_i(\zeta) = 0$ for all i. If we write $P_j = \sum P_{i,j}\beta_i$, $Q = \sum Q_i \beta_i$ then there is some i_0 such that η satisfies $\{P_{i,j} = 0\}$, $Q_{i_0} \neq 0$ and that for any $\zeta \in C_E^\Lambda$ that satisfies this system, the map $\eta \mapsto \zeta$ induces a Δ isomorphism of $k\langle\eta\rangle_\Delta$ and $k\langle\zeta\rangle_\Delta$.

We therefore may assume that there exist $P_1, \dots, P_s, Q \in C_k^\Lambda\{y\}_\Delta$ satisfying the conditions of Definition 9.1 with respect to η and k. We now show that there exist $\tilde{P}_1, \dots, \tilde{P}_s, \tilde{Q}$ in the smaller differential polynomial ring $C_k^\Lambda\{y\}_\Pi$ satisfying: If E is a Δ-extension of k and $\zeta \in C_E^\Lambda$ satisfies $\tilde{P}_1(\zeta) = \dots = \tilde{P}_s(\zeta) = 0$, $\tilde{Q}(\zeta) \neq 0$ then there is a k-Δ-isomorphism of $k\langle\eta\rangle_\Delta$ and $k\langle\zeta\rangle_\Delta$ mapping $\eta \mapsto \zeta$. To do this, note that any $P \in C_k^\Lambda\{y\}_\Delta$ is a C_k^Λ-linear combination of monomials that are products of terms of the form $\partial_0^{i_0} \dots \partial_m^{i_m} y$. We denote by $\tilde{P}$ the differential polynomial resulting from P be deleting any monomial that contains a term $\partial_0^{i_0} \dots \partial_m^{i_m} y_j$ *with* $i_t > 0$ *for some* $\partial_{i_t} \in \Lambda$. Note that for any Δ-extension E of k and $\zeta \in C_E^\Lambda$ we have $P(\zeta) = 0$ if and only if $\tilde{P}(\zeta) = 0$. Therefore, for any $\zeta \in C_E^\Lambda$, if $\tilde{P}_1(\zeta) = \dots = \tilde{P}_1(\zeta) = 0$ and $\tilde{Q}(\zeta) \neq 0$, then the map $\eta \mapsto \zeta$ induces a Δ-k-isomorphism of $k\{\eta\}_\Delta$ with $k\{\zeta\}_\Delta$.

We now use the fact that C_k^Λ is a Π-differentially closed field to show that any $\eta \in C_K^\Lambda$ must already be in C_k^Λ. Let $\tilde{P}_1, \dots, \tilde{P}_s, \tilde{Q} \in C_k^\Lambda\{y\}_\Pi$ be as above. Since C_k^Λ is a Π-differentially closed field and $\tilde{P}_1 = \dots = \tilde{P}_s = 0$, $\tilde{Q} \neq 0$ has a solution in *some* Π-extension of C_k^Λ (*e.g.*, $\eta \in C_K^\Lambda$), this system has a solution $\zeta \in C_k^\Lambda \subset k$. We therefore can conclude that the map $\eta \mapsto \zeta$ induces a Π-k-isomorphism from $k\langle\eta\rangle$ to $k\langle\zeta\rangle$. Since $\zeta \in k$, we have that $\eta \in k$ and so $\eta \in C_k^\Lambda$. □

We note that if Π is empty, then Π-differentially closed is the same as algebraically closed. In this case the above result yields the important fact crucial to the Picard–Vessiot theory mentioned before the lemma.

9.2 PPV-extensions

In the next three sections, we will develop the theory of PPV-extensions for parameterized integrable systems of linear differential equations. This section is devoted to showing the existence and uniqueness of these extensions. In Section 9.3 we show that the Galois group has a natural structure as a linear differential algebraic group and in Section 9.4 we show that a PPV-extension can be associated with a torsor for the Galois group. As in the usual Picard–Vessiot theory, these results will allow us to give a complete Galois theory (see Theorem 9.5).

In this and the next three sections, we will make the following conventions. We let k be a Δ-differential field. We designate a nonempty subset $\Lambda = \{\partial_0, \dots, \partial_r\} \subset \Delta$ and consider a system of linear differential equations

$$\begin{aligned} \partial_0 Y &= A_0 Y \\ \partial_1 Y &= A_1 Y \\ &\vdots \\ \partial_r Y &= A_r Y \end{aligned} \tag{9.1}$$

where the $A_i \in \mathrm{gl}_n(k)$, the set of $n \times n$ matrices with entries in k, such that

$$\partial_i A_j - \partial_j A_i = [A_i, A_j] \tag{9.2}$$

We denote by Π the set $\Delta \backslash \Lambda$. One sees that the derivations of Π leave the field C_k^Λ invariant and we shall think of this latter field as a Π-field. Throughout the next sections, we shall assume that $C = C_k^\Lambda$ *is a* Π*-differentially closed differential field.* The set Λ corresponds to derivations used in the linear differential equations and Π corresponds to the parametric derivations.Throughout the first part of this paper Δ was $\{\partial_0, \dots, \partial_m\}$, $\Lambda = \{\partial_0\}$, and $\Pi = \{\partial_1, \dots, \partial_m\}$. We now turn to a definition.

Definition 9.4. (1) A *parameterized Picard–Vessiot ring* (PPV-ring) over k for the equations (9.1) is a Δ-ring R containing k satisfying:

(a) R is a Δ-simple Δ-ring.

(b) There exists a matrix $Z \in \mathrm{GL}_n(R)$ such that $\partial_i Z = A_i Z$ for all $\partial_i \in \Lambda$.

(c) R is generated, as a Δ-ring over k, by the entries of Z and $1/\det(Z)$, *i.e.*, $R = k\{Z, 1/\det(Z)\}_\Delta$.

(2) A *parameterized Picard–Vessiot extension of* k (PPV-extension of k) for the equations (9.1) is a Δ-field K satisfying

(a) $k \subset K$.

(b) There exists a matrix $Z \in \mathrm{GL}_n(K)$ such that $\partial_i Z = A_i Z$ for all $\partial_i \in \Lambda$ and K is generated as a Δ-field over k by the entries of Z.

(c) $C_K^\Lambda = C_k^\Lambda$, *i.e.,* the Λ-constants of K coincide with the Λ-constants of k.

(3) The group $\mathrm{Gal}_\Delta(K/k) = \{\sigma : K \to K \mid \sigma$ is a k-automorphism such that $\sigma\partial = \partial\sigma$ for all $\partial \in \Delta\}$ is called the *parameterized Picard–Vessiot group (PPV-group)* associated with the PPV-extension K of k.

Note that when $\Delta = \Lambda$, $\Pi = \emptyset$ these definitions give us the corresponding definitions in the usual Picard–Vessiot theory.

Our goal in the next three sections is to prove results that will yield the following generalization of both Theorem 2.1 (when $\Delta = \Lambda$) and Theorem 3.5 (when $\Delta = \{\partial_0, \partial_1, \ldots, \partial_m\}$ and $\Lambda = \{\partial_0\}$).

Theorem 9.5. (1) *There exists a PPV-extension K of k associated with* (9.1) *and this is unique up to Δ-k-isomorphism.*

(2) *The PPV-group* $\mathrm{Gal}_\Delta(K/k)$ *equals $G(C_k^\Lambda)$, where G is a linear Π-differential algebraic group defined over C_k^Λ.*

(3) *The map that sends any Δ-subfield F, $k \subset F \subset K$, to the group* $\mathrm{Gal}_\Delta(K/F)$ *is a bijection between Δ-subfields of K containing k and Π-Kolchin closed subgroups of* $\mathrm{Gal}_\Delta(K/k)$. *Its inverse is given by the map that sends a Π-Kolchin closed group H to the field $\{z \in K \mid \sigma(z) = z$ for all $\sigma \in H\}$.*

(4) *A Π-Kolchin closed subgroup H of* $\mathrm{Gal}_\Delta(K/k)$ *is a normal subgroup of* $\mathrm{Gal}_\Delta(K/k)$ *if and only if the field K^H is left set-wise invariant by* $\mathrm{Gal}_\Delta(K/k)$. *If this is the case, the map* $\mathrm{Gal}_\Delta(K/k) \to \mathrm{Gal}_\Delta(K^H/k)$ *is surjective with kernel H and K^H is a PPV-extension of k with PPV-group isomorphic to* $\mathrm{Gal}_\Delta(K/k)/H$. *Conversely, if F is a differential subfield of K containing k and F is a PPV-extension of k, then* $\mathrm{Gal}_\Delta(K/F)$ *is a normal Π-Kolchin closed subgroup of* $\mathrm{Gal}_\Delta(K/k)$.

We shall show in this section that PPV-rings for (9.1) exist and are unique up to Δ-k-isomorphism and that every PPV-extension K of k is the quotient field of a PPV-ring (and therefore is also unique up to Δ-k-isomorphism.) We begin with

Proposition 9.6. (1) *There exists a PPV-ring R for* (9.1) *and it is an integral domain.*

(2) *The field of Λ-constants C_K^Λ of the quotient field K of a PPV-ring over k is C_k^Λ.*

(3) *Any two PPV-rings for this system are k-isomorphic as Δ-rings.*

Proof. (1) Let $(Y_{i,j})$ denote an $n \times n$ matrix of Π-indeterminates and let "det" denote the determinant of $(Y_{i,j})$. We denote by $k\{Y_{1,1}, \ldots, Y_{n.n}, 1/\det\}_\Pi$ the Π-differential polynomial ring in the variables $\{Y_{i,j}\}$ localized at det. We can make this ring into a Δ-ring by setting $(\partial_k Y_{i,j}) = A_k(Y_{i,j})$ for all $\partial_k \in \Lambda$ and using the fact that $\partial_k\partial_l = \partial_l\partial_k$ for all $\partial_k, \partial_l \in \Delta$. Let p be a maximal Δ-ideal in R. One then sees that R/p is a PPV-ring for the equation. Since maximal differential ideals are prime,R is an integral domain.

(2) Let $R = k\{Z, 1/\det(Z)\}_\Delta$. Since this is a simple differential ring, Proposition 9.2 (1) implies that Z is constrained over k with constraint det. Statement (2) of Proposition 9.2 implies that the quotient field of R is a Δ-constrained extension of k. Lemma 9.3 implies that $C_K^\Lambda = C_k^\Lambda$.

(3) Let R_1, R_2 denote two PPV-rings for the system. Let Z_1, Z_2 be the two fundamental matrices. Consider the Δ-ring $R_1 \otimes_k R_2$ with derivations $\partial_i(r_1 \otimes r_2) = \partial_i r_1 \otimes r_2 + r_1 \otimes \partial_i r_2$. Let p be a maximal Δ-ideal in $R_1 \otimes_k R_2$ and let $R_3 = R_1 \otimes_k R_2/p$. The obvious maps $\phi_i : R_i \to R_1 \otimes_k R_2$ are Δ-homomorphisms and, since the R_i are

simple, the homomorphisms ϕ_i are injective. The image of each ϕ_i is differentially generated by the entries of $\phi_i(Z_i)$ and $\det(\phi(Z_i^{-1}))$. The matrices $\phi_1(Z_1)$ and $\phi_2(Z_2)$ are fundamental matrices in R_3 of the differential equation. Since R_3 is simple, the previous result implies that C_k^Λ is the ring of Λ-constants of R_3. Therefore $\phi_1(Z_1) = \phi_2(Z_2)D$ for some $D \in \mathrm{GL}_n(C_k^\Lambda)$. Therefore $\phi_1(R_1) = \phi_2(R_2)$ and so R_1 and R_2 are isomorphic. □

Conclusion (2) of the above proposition shows that the field of fractions of a PPV-ring is a PPV-field. We now show that a PPV-field for an equation is the field of fractions of a PPV-ring for the equation.

Proposition 9.7. *Let K be a PPV-extension field of k for the system* (9.1)*, let $Z \in \mathrm{GL}_n(K)$ satisfy $\partial_i(Z) = A_i Z$ for all $\partial_i \in \Lambda$ and let* $\det = \det(Z)$.

(1) *The Δ-ring $k\{Z, 1/\det\}_\Delta$ is a PPV-extension ring of k for this system.*
(2) *If K' is another PPV-extension of k for this system then there is a k-Δ-isomorphism of K and K'.*

To simplify notation we shall use $\frac{1}{\det}$ to denote the inverse of the determinant of a matrix given by the context. For example, $k\left\{Y_{i,j}, \frac{1}{\det}\right\}_\Delta = k\left\{Y_{i,j}, \frac{1}{\det(Y_{i,j})}\right\}_\Delta$ and $k\left\{X_{i,j}, \frac{1}{\det}\right\}_\Pi = k\left\{X_{i,j}, \frac{1}{\det(X_{i,j})}\right\}_\Pi$.

As in [40], p. 16, we need a preliminary lemma to prove this proposition. Let $(Y_{i,j})$ be an $n \times n$ matrix of Π-differential indeterminates and let det denote the determinant of this matrix. For any Π-field k, we denote by $k\{Y_{i,j}, 1/\det\}_\Pi$ the Π-ring of differential polynomials in the $Y_{i,j}$ localized with respect to det. If k is, in addition, a Δ-field, the derivations $\partial \in \Lambda$ can be extended to $k\{Y_{i,j}, 1/\det\}_\Pi$ by setting $\partial(Y_{i,j}) = 0$ for all $\partial \in \Lambda$ and i, j with $1 \le i, j \le n$. In this way $k\{Y_{i,j}, 1/\det\}_\Pi$ may be considered as a Δ-ring. We consider $C_k^\Lambda\{Y_{i,j}, 1/\det\}_\Pi$ as a Π-subring of $k\{Y_{i,j}, 1/\det\}_\Pi$. For any set $I \subset k\{Y_{i,j}, 1/\det\}_\Pi$, we denote by $(I)_\Delta$ the Δ-differential ideal in $k\{Y_{i,j}, 1/\det\}_\Pi$ generated by I.

Lemma 9.8. *Using the above notation, the map $I \mapsto (I)_\Delta$ from the set of Π-ideals of $C_k^\Lambda\{Y_{i,j}, 1/\det\}_\Pi$ to the set of Δ- ideals of $k\{Y_{i,j}, 1/\det\}_\Pi$ is a bijection. The inverse map is given by $J \mapsto J \cap C_k^\Lambda\{Y_{i,j}, 1/\det\}_\Pi$.*

Proof. If $\mathcal{S} = \{s_\alpha\}_{\alpha\in\mathrm{A}}$ is a basis of k over C_k^Λ, then $\mathcal{S}$ is a module basis of $k\left\{Y_{i,j}, \frac{1}{\det}\right\}_\Pi$ over $C_k^\Lambda\left\{Y_{i,j}, \frac{1}{\det}\right\}_\Pi$. Therefore, for any ideal I of $C_k^\Lambda\left\{Y_{i,j}, \frac{1}{\det}\right\}_\Pi$, one has that $(I)_\Delta \cap C_k^\Lambda\left\{Y_{i,j}, \frac{1}{\det}\right\}_\Pi = I$.

We now prove that any Δ-differential ideal J of $k\left\{Y_{i,j}, \frac{1}{\det}\right\}_\Pi$ is generated by $I := J \cap C_k^\Lambda\left\{Y_{i,j}, \frac{1}{\det}\right\}_\Pi$. Let $\{e_\beta\}_{\beta\in\mathcal{B}}$ be a basis of $C_k^\Lambda\left\{Y_{i,j}, \frac{1}{\det}\right\}_\Pi$ over C_k^Λ. Any element $f \in J$ can be uniquely written as a finite sum $\sum_{\beta\in\mathcal{B}} m_\beta e_\beta$ with the $m_\beta \in k$. By induction on the length, $l(f)$, of f we will show that $f \in (I)_\Delta$. When $l(f) = 0, 1$, the result is clear. Assume $l(f) > 1$. We may suppose that $m_{\beta_1} = 1$ for some

$\beta_1 \in \mathcal{B}$ and $m_{\beta_2} \in k\backslash C_k^\Lambda$ for some $\beta_2 \in \mathcal{B}$. One then has that, for any $\partial \in \Lambda$, $\partial f = \sum_\beta \partial m_\beta e_\beta$ has a length smaller than $l(f)$ and so belongs to $(I)_\Delta$. Similarly $\partial(m_{\beta_2}^{-1} f) \in (I)_\Delta$. Therefore $\partial(m_{\beta_2}^{-1})f = \partial(m_{\beta_2}^{-1} f) - m_{\beta_2}^{-1}\partial f \in (I)_\Delta$. Since C_k^Λ is the field of Λ-constants of k, one has $\partial_i(m_{\beta_2}^{-1}) \neq 0$ for some $\partial_i \in \Lambda$ and so $f \in (I)_\Delta$. □

Proof of Proposition 9.7. (1) Let $R_0 = k\{X_{i,j}, \frac{1}{\det}\}_\Pi$ be the ring of Π-differential polynomials over k and define a Δ-structure on this ring by setting $(\partial_i X_{i,j}) = A_i(X_{i,j})$ for all $\partial_i \in \Lambda$. Consider the Δ-rings $R_0 \subset K \otimes_k R_0 = K\{X_{i,j}, \frac{1}{\det}\}_\Pi$. Define a set of n^2 new variables $Y_{i,j}$ by $(X_{i,j}) = Z \cdot (Y_{i,j})$. Then $K \otimes_k R_0 = K\{Y_{i,j}, \frac{1}{\det}\}_\Pi$ and $\partial Y_{i,j} = 0$ for all $\partial \in \Lambda$ and all i, j. We can identify $K \otimes_k R_0$ with $K \otimes_{C_k^\Lambda} R_1$ where $R_1 := C_k^\Lambda\{Y_{i,j}, \frac{1}{\det}\}_\Pi$. Let P be a maximal Δ-ideal of R_0. P generates an ideal in $K \otimes_k R_0$ which is denoted by (P). Since $K \otimes R_0/(P) \cong K \otimes (R_0/P) \neq 0$, the ideal (P) is a proper differential ideal. Define the ideal $\tilde{P} \subset R_1$ by $\tilde{P} = (P) \cap R_1$. By Lemma 9.8 the ideal (P) is generated by $\tilde{P}$. If M is a maximal Π-ideal of R_1 containing $\tilde{P}$ then R_1/M is a simple, finitely generated Π-extension of C_k^Λ and so is a constrained extension of C_k^Λ. Since C_k^Λ is differentially closed, Proposition 9.2 (3) implies that $R_1/M = C_k^\Lambda$. The corresponding homomorphism of C_k^Λ-algebras $R_1 \to C_k^\Lambda$ extends to a differential homomorphism of K-algebras $K \otimes_{C_k^\Lambda} R_1 \to K$. Its kernel contains $(P) \subset K \otimes_k R_0 = K \otimes_{C_k^\Lambda} R_1$. Thus we have found a k-linear differential homomorphism $\psi\colon R_0 \to K$ with $P \subset \ker(\psi)$. The kernel of ψ is a differential ideal and so $P = \ker(\psi)$. The subring $\psi(R_0) \subset K$ is isomorphic to R_0/P and is therefore a PPV-ring. The matrix $(\psi(X_{i,j}))$ is a fundamental matrix in $\mathrm{GL}_n(K)$ and must have the form $Z \cdot (c_{i,j})$ with $(c_{i,j}) \in \mathrm{GL}_n(C_k^\Lambda)$, because the field of Λ-constants of K is C_k^Λ. Therefore, $k\{Z, 1/\det\}_\Delta$ is a PPV-extension of k.

(2) Let K' be a PPV-extension of k for $\partial_0 Y = AY$. Part (1) of this proposition implies that both K' and K are quotient fields of PPV-rings for this equation. Proposition 9.6 implies that these PPV-rings are k-Δ-isomorphic and the conclusion follows. □

The following result was used in Proposition 3.9.

Lemma 9.9. *Let* $\Delta = \{\partial_0, \partial_1, \ldots, \partial_m\}$ *and* $\Lambda = \{\partial_0\}$. *Let*

$$\begin{aligned} \partial_0 Y &= AY \\ \partial_1 Y &= A_1 Y \\ &\vdots \\ \partial_m Y &= A_m Y \end{aligned} \tag{9.3}$$

be an integrable system with $A_i \in \mathrm{gl}_n(k)$. *If* K *is a PV-extension of* k *for* (9.3), *then* K *is a PPV-extension of* k *for* $\partial_0 Y = AY$

Proof. We first note that C_k^{Δ} is a subfield of C_k^{Λ}. Since this latter field is differentially closed, it is algebraically closed. Therefore, C_k^{Δ} is also algebraically closed. The usual Picard–Vessiot theory[6] implies that K is the quotient field of the Picard–Vessiot ring $R = k\{Z, 1/\det Z\}_{\Delta}$ where Z satisfies the system (9.3). Since R is a simple Δ-ring, we have that Z is constrained over k, Proposition 9.2 (2) implies that K is a Δ-constrained extension of k. Since C_k^{Λ} is differentially closed, Lemma 9.3 implies that $C_K^{\partial_0} = C_k^{\partial_0}$ so K is a PPV-extension of k. □

9.3 Galois groups

In this section we shall show that the PPV-group $\mathrm{Gal}_{\Delta}(K/k)$ of a PPV-extension K of k is a linear differential algebraic group and also show the correspondence between Kolchin-closed subgroups of $\mathrm{Gal}_{\Delta}(K/k)$ and Δ-subfields of K containing k. This is done in the next Proposition and conclusions (2) and (3) of Theorem 3.5 are immediate consequences.

To make things a little more precise, we will use a little of the language of affine differential algebraic geometry (see [9] or [22] for more details). We begin with some definitions that are the obvious differential counterparts of the usual definitions in affine algebraic geometry. Let k be a Δ-field. An *affine differential variety* V defined over k is given by a radical differential ideal $I \subset k\{Y_1, \ldots, Y_n\}_{\Delta}$. In this case, we shall say V is a differential subvariety of affine n-space and write $V \subset \mathbb{A}^n$. We will identify V with its *coordinate ring* $k\{V\} = k\{Y_1, \ldots, Y_n\}_{\Delta}/I$. Conversely, given a reduced Δ-ring R that is finitely generated (in the differential sense) as a k-algebra, we may associate with it the differential variety V defined by the radical differential ideal I where $R = k\{Y_1, \ldots, Y_n\}_{\Delta}/I$. Given any Δ-field $K \supset k$, the set of K-points of V, denoted by $V(K)$, is the set of points of K^n that are zeroes of the defining ideal of V, and may be identified with the set of k-Δ-homomorphisms of $k\{V\}$ to K. If $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^p$ are affine differential varieties defined over k, a *differential polynomial map* $f\colon V \to W$ is given by a p-tuple $(f_1, \ldots, f_p) \in (k\{Y_1, \ldots, Y_n\}_{\Delta})^p$ such that the map that sends an $F \in k\{Y_1, \ldots, Y_p\}_{\Delta}$ to $F(f_1, \ldots, f_p) \in k\{Y_1, \ldots, Y_n\}_{\Delta}$ induces a k-Δ-homomorphism f^* of $k\{W\}$ to $k\{V\}$. A useful criterion for showing that a p-tuple $(f_1, \ldots, f_p) \in (k\{Y_1, \ldots, Y_n\}_{\Delta})^p$ defines a differential polynomial map from V to W is the following: $(f_1, \ldots, f_p)$ *defines a differential polynomial map from* V *to* W *if and only if for any* Δ*-field* $K \supset k$ *and any* $v \in V(K)$*, we have* $(f_1(v), \ldots, f_p(v)) \in W(K)$. This is an easy consequence of the *theorem of zeros* ([21], Ch. IV.2) which in turn is an easy consequence of the fact that a radical differential ideal is the intersection of prime differential ideals.

Given affine differential varieties V and W defined over k, we define the *product* $V \times_k W$ *of* V *and* W to be the differential affine variety associated with $k\{V\} \otimes_k k\{W\}$. Note that since our fields have characteristic zero, this latter ring is reduced.

[6] Proposition 1.22 of [40] proves this only for the ordinary case. Proposition 9.7 above yields this result if we let $\Lambda = \Delta$.

In this setting, a linear differential algebraic group G (defined over k) is the affine differential algebraic variety associated with a radical differential ideal $I \subset k\{Y_{1,1,}, \ldots, Y_{n,n}, Z\}_\Delta$ such that

(1) $1 - Z \cdot \det((Y_{i,j})) \in I$,

(2) $(\mathrm{id}, 1) \in G(k)$ where id is the $n \times n$ identity matrix.

(3) the map given by matrix multiplication

$$(g, (\det g)^{-1})(h, (\det h)^{-1}) \mapsto (gh, (\det(gh))^{-1})$$

(which is obviously a differential polynomial map) is a map from $G \times G$ to G and the inverse map $(g, (\det g)^{-1}) \mapsto (g^{-1}, \det g)$ (also a differential polynomial map) is a map from G to G.

Since we assume that $1 - Z \cdot \det((Y_{i,j})) \in I$, we may assume that G is defined by a radical differential ideal in the ring $k\{Y_{1,1,}, \ldots, Y_{n,n}, 1/\det(Y_{i,j})\}_\Delta$, which we abbreviate as $k\{Y, 1/\det Y\}_\Delta$. In this way, for any $K \supset k$ we may identify $G(K)$ with elements of $\mathrm{GL}_n(K)$ and the multiplication and inversion is given by the usual operations on matrices. We also note that the usual Hopf algebra definition of a linear algebraic group carries over to this setting as well. See [10] for a discussion of k-differential Hopf algebras, and criteria for an affine differential algebraic group to be linear.

Proposition 9.10. *Let $K \supset k$ be a PPV-field with differential Galois group* $\mathrm{Gal}_\Delta(K/k)$. *Then*

(1) $\mathrm{Gal}_\Delta(K/k)$ *is the group of C_k^Λ-points $G(C_k^\Lambda) \subset \mathrm{GL}_n(C_k^\Lambda)$ of a linear Π-differential algebraic group G over C_k^Λ.*

(2) *Let H be a subgroup of* $\mathrm{Gal}_\Delta(K/k)$ *satisfying $K^H = k$. Then the Kolchin closure $\bar{H}$ of H is* $\mathrm{Gal}_\Delta(K/k)$.

(3) *The field $K^{\mathrm{Gal}_\Delta(K/k)}$ of* $\mathrm{Gal}_\Delta(K/k)$*-invariant elements of the Picard–Vessiot field K is equal to k.*

Proof. (1) We shall show that there is a radical Π-ideal $I \subset S = C_k^\Lambda\left\{Y_{i,j}, \frac{1}{\det}\right\}_\Pi$ such that S/I is the coordinate ring of a linear Π-differential algebraic group G and $\mathrm{Gal}_\Delta(K/k)$ corresponds to $G(C_k^\Lambda)$.

Let K be the PPV-extension for the integrable system (9.1). Once again we denote by $k\left\{X_{i,j}, \frac{1}{\det}\right\}_\Pi$ the Π-differential polynomial ring with the added Δ-structure defined by $(\partial_r X_{i,j}) = A_r(X_{i,j})$ for $\partial_r \in \Lambda$. K is the field of fractions of $R := k\left\{X_{i,j}, \frac{1}{\det}\right\}_\Pi/q$, where q is a maximal Δ-ideal. Let $r_{i,j}$ be the image of $X_{i,j}$ in R so $(r_{i,j})$ is a fundamental matrix for the equations $\partial_i Y = A_i Y$, $\partial_i \in \Lambda$. Consider the following rings:

$$k\left\{X_{i,j}, \tfrac{1}{\det}\right\}_\Pi \subset K\left\{X_{i,j}, \tfrac{1}{\det}\right\}_\Pi = K\left\{Y_{i,j}, \tfrac{1}{\det}\right\}_\Pi \supset C_k^\Lambda\left\{Y_{i,j}, \tfrac{1}{\det}\right\}_\Pi$$

where the indeterminates $Y_{i,j}$ are defined by $(X_{i,j}) = (r_{i,j})(Y_{i,j})$. Note that $\partial Y_{i,j} = 0$ for all $\partial \in \Pi$. Since all fields are of characteristic zero, the ideal $qK\left\{Y_{i,j}, \frac{1}{\det}\right\}_\Pi \subset$

$K\{X_{i,j}, \frac{1}{\det}\}_\Pi = K\{Y_{i,j}, \frac{1}{\det}\}_\Pi$ is a radical Δ-ideal (*cf.,* [40], Corollary A.16). It follows from Lemma 9.8 that $qL[Y_{i,j}, \frac{1}{\det}]$ is generated by $I = qK\{Y_{i,j}, \frac{1}{\det}\}_\Pi \cap C_k^\Lambda\{Y_{i,j}, \frac{1}{\det}\}_\Pi$. Clearly I is a radical Δ-ideal of $S = C_k^\Lambda\{Y_{i,j}, \frac{1}{\det}\}_\Pi$. We shall show that S/I is the Π-coordinate ring of a linear differential algebraic group G, inheriting its group structure from GL_n. In particular, we shall show that $G(C_k^\Lambda)$ is a subgroup of $\mathrm{GL}_n(C_k^\Lambda)$ and that there is an isomorphism of $\mathrm{Gal}_\Delta(K/k)$ onto $G(C_k^\Lambda)$.

$\mathrm{Gal}_\Delta(K/k)$ can be identified with the set of $(c_{i,j}) \in \mathrm{GL}_n(C_k^\Lambda)$ such that the map $(X_{i,j}) \mapsto (X_{i,j})(c_{i,j})$ leaves the ideal q invariant. One can easily show that the following statements are equivalent:

(i) $(c_{i,j}) \in \mathrm{Gal}_\Delta(K/k)$,

(ii) The map $k\{X_{i,j}, \frac{1}{\det}\}_\Pi \to K$ defined by $(X_{i,j}) \mapsto (r_{i,j})(c_{i,j})$ maps all elements of q to zero.

(iii) The map $K\{X_{i,j}, \frac{1}{\det}\}_\Pi \to K$ defined by $(X_{i,j}) \mapsto (r_{i,j})(c_{i,j})$ maps all elements of $qK\{X_{i,j}, \frac{1}{\det}\}_\Pi = qK\{Y_{i,j}, \frac{1}{\det}\}_\Pi$ to zero.

(iv) Considering $qK\{Y_{i,j}, \frac{1}{\det}\}_\Pi$ as an ideal of $K\{X_{i,j}, \frac{1}{\det}\}_\Pi$, the map

$$K\{Y_{i,j}, \tfrac{1}{\det}\}_\Pi \to K, \quad (Y_{i,j}) \mapsto (c_{i,j}),$$

sends all elements of $qK\{Y_{i,j}, \frac{1}{\det}\}_\Pi$ to zero.

Since the ideal $qK\{Y_{i,j}, \frac{1}{\det}\}_\Pi$ is generated by I, the last statement above is equivalent to $(c_{i,j})$ being a zero of the ideal I, i.e., $(c_{i,j}) \in G(C_k^\Lambda)$. Since $\mathrm{Gal}_\Delta(K/k)$ is a group, the set $G(C_k^\Lambda)$ is a subgroup of $\mathrm{GL}_n(C_k^\Lambda)$. Therefore G is a linear differential algebraic group.

(2) Assuming that $\bar{H} \neq \mathrm{Gal}_\Delta$, we shall derive a contradiction. We shall use the notation of part (1) above. If $\bar{H} \neq \mathrm{Gal}_\Delta$, then there exists an element $P \in C_k^\Lambda\{Y_{i,j}, \frac{1}{\det}\}_\Pi$ such that $P \notin I$ and $P(h) = 0$ for all $h \in H$. Lemma 9.8 implies that $P \notin (I) = qk\{Y_{i,j}, \frac{1}{\det}\}_\Pi$. Let $T = \{Q \in K\{X_{i,j}, \frac{1}{\det}\}_\Pi \mid Q \notin (I)$ and $Q((r_{i,j})(h_{i,j})) = 0$ for all $h = (h_{i,j}) \in H\}$. Since $K\{X_{i,j}, \frac{1}{\det}\}_\Pi = K\{Y_{i,j}, \frac{1}{\det}\}_\Pi \supset C_k^\Lambda\{Y_{i,j}, \frac{1}{\det}\}_\Pi$ we have that $T \neq \{0\}$. Any element of $K\{X_{i,j}, \frac{1}{\det}\}_\Pi$ may be written as $\sum_\alpha f_\alpha Q_\alpha$ where $f_\alpha \in K$ and $Q_\alpha \in k\{X_{i,j}, \frac{1}{\det}\}_\Pi$. Select $Q = f_{\alpha_1}Q_{\alpha_1} + \cdots + f_{\alpha_m}Q_{\alpha_m} \in T$ with the f_{α_i} all nonzero and m minimal. We may assume that $f_{\alpha_1} = 1$. For each $h \in H$, let $Q^h = f_{\alpha_1}^h Q_{\alpha_1} + \cdots + f_{\alpha_m}^h Q_{\alpha_m}$. One sees that $Q^h \in T$. Since $Q - Q^h$ is shorter than Q and satisfies $(Q - Q^h)((r_{i,j})(h_{i,j})) = 0$ for all $h = (h_{i,j}) \in H$ we must have that $Q - Q^h \in (I)$. If $Q - Q^h \neq 0$ then there exists an $l \in K$ such that $Q - l(Q - Q^h)$ is shorter than Q. One sees that $Q - l(Q - Q^h) \in T$ yielding a contradiction unless $Q - Q^h = 0$. Therefore $Q = Q^h$ for all $h \in H$ and so the $f_{\alpha_i} \in k$. We conclude that $Q \in k\{X_{i,j}, \frac{1}{\det}\}_\Pi$. Since $Q(r_{i,j}) = 0$ we have that $Q \in q$, a contradiction.

(3) Let $a = \frac{b}{c} \in K\backslash k$ with $b, c \in R$ and $d = b \otimes c - c \otimes b \in R \otimes_k R$. Elementary properties of tensor products imply that $d \neq 0$ since b and c are linearly independent over C_k^Λ. The ring $R \otimes_k R$ has no nilpotent elements since the characteristic of k is zero

(*cf.,* [40], Lemma A.16). We define a Δ-ring structure on $R\otimes_k R$ by letting $\partial(r_1\otimes r_2) = \partial(r_1)\otimes r_2 + r_1\otimes\partial(r_2)$ for all $\partial\in\Delta$. Let J be a maximal differential ideal in the differential ring $(R\otimes_k R)\left[\frac{1}{d}\right]$. Consider the two obvious morphisms $\phi_i\colon R\to N := (R\otimes_k R)\left[\frac{1}{d}\right]/J$. The images of the ϕ_i are generated (over k) by fundamental matrices of the same matrix differential equation. Therefore both images are equal to a certain subring $S\subset N$ and the maps $\phi_i\colon R\to S$ are isomorphisms. This induces an element $\sigma\in G$ with $\phi_1=\phi_2\sigma$. The image of d in N is equal to $\phi_1(b)\phi_2(c)-\phi_1(c)\phi_2(b)$. Since the image of d in N is nonzero, one finds $\phi_1(b)\phi_2(c)\neq\phi_1(c)\phi_2(b)$. Therefore $\phi_2((\sigma b)c)\neq\phi_2((\sigma c)b)$ and so $(\sigma b)c\neq(\sigma c)b$. This implies $\sigma\left(\frac{b}{c}\right)\neq\frac{b}{c}$. □

We have therefore completed proof of parts (2) and (3) of Theorem 9.5.

9.4 PPV-rings and torsors

In this section we will prove conclusion (4) of Theorem 9.5. As in the usual Picard–Vessiot theory, this depends on identifying the PPV-extension ring as the coordinate ring of a torsor of the PPV-group.

Definition 9.11. Let k be a Π-field and G a linear differential algebraic group defined over k. A *G-torsor (defined over k)* is an affine differential algebraic variety V defined over k together with a differential polynomial map $f\colon V\times_k G\to V\times_k V$ (denoted by $f\colon(v,g)\mapsto(vg,v)$) such that

(1) for any Π-field $K\supset k$, $v\in V(K)$, $g,g_1,g_2\in G(K)$, $v1_G=v$, $v(g_1g_2)=(vg_1)g_2$ and

(2) the associated homomorphism $k\{V\}\otimes_k k\{V\}\to k\{V\}\otimes_k k\{G\}$ is an isomorphism (or equivalently, for any $K\supset k$, the map $V(K)\times G(K)\to V(K)\times V(K)$ is a bijection.

We note that $V=G$ is a torsor for G over k with the action given by multiplication. This torsor is called the *trivial torsor over k*. We shall use the following notation. If V is a differential affine variety defined over k with coordinate ring $R=k\{V\}$ and $K\supset k$ we denote by V_K the differential algebraic variety (over K) whose coordinate ring is $R\otimes_k K=K\{V\}$.

We again consider the integrable system (9.1) over the Δ-field k. The PPV-ring for this equation has the form $R=k\left\{X_{i,j},\frac{1}{\det}\right\}_\Pi/q$, where q is a maximal Δ-ideal. In the following, we shall think of q as only a Π-differential ideal. We recall that $k\left\{X_{i,j},\frac{1}{\det}\right\}_\Pi$ is the coordinate ring of the linear Π-differential algebraic group GL_n over k. Let V be the affine differential algebraic variety associated with the ring $k\left\{X_{i,j},\frac{1}{\det}\right\}_\Pi/q$. This is an irreducible and reduced Π-Kolchin-closed subset of GL_n. Let K denote the field of fractions of $k\left\{X_{i,j},\frac{1}{\det}\right\}_\Pi/q$. We have shown in the previous section that the PPV-group $\mathrm{Gal}_\Delta(K/k)$ of this equation may be identified with $G(C_k^\Lambda)$, that is the C_k^Λ-points of a Π-linear differential algebraic group G over C_k^Λ. We recall how G was defined. Consider the following rings

$$k\left\{X_{i,j}, \tfrac{1}{\det}\right\}_\Pi \subset K\left\{X_{i,j}, \tfrac{1}{\det}\right\}_\Pi = K\left\{Y_{i,j}, \tfrac{1}{\det}\right\}_\Pi \supset C_k^\Lambda\left\{Y_{i,j}, \tfrac{1}{\det}\right\}_\Pi,$$

where the relation between the variables $X_{i,j}$ and the variables $Y_{i,j}$ is given by $(X_{i,j}) = (r_{i,j})(Y_{i,j})$. The $r_{a,b} \in K$ are the images of $X_{a,b}$ in $k\left\{X_{i,j}, \frac{1}{\det}\right\}_\Pi/q \subset K$. In Proposition 9.10 we showed that the ideal $I = qK\left\{X_{i,j}, \frac{1}{\det}\right\}_\Pi \cap C_k^\Lambda\left\{Y_{i,j}, \frac{1}{\det}\right\}_\Pi$ defines G. This observation is the key to showing the following.

Proposition 9.12. *V is a G-torsor over k.*

Proof. Let E be a Δ-field containing k. The group $G(C_k^\Lambda) \subset \mathrm{GL}_n(C_k^\Lambda)$ is precisely the set of matrices $(c_{i,j})$ such that the map $(X_{i,j}) \mapsto (X_{i,j})(c_{i,j})$ leaves the ideal q stable. In particular, for $(c_{i,j}) \in G(C_k^\Lambda)$, $(\bar{z}_{i,j}) \in V(E)$ we have that $(\bar{z}_{i,j})(c_{i,j}) \in V(E)$. We will first show that this map defines a morphism from $V \times G_k \to V$. The map is clearly defined over k so we need only show that for any $(\bar{c}_{i,j}) \in G(E)$, $(\bar{z}_{i,j}) \in V(E)$ we have that $(\bar{z}_{i,j})(\bar{c}_{i,j}) \in V(E)$. Assume that this is not true and let $(\bar{c}_{i,j}) \in G(E)$, $(\bar{z}_{i,j}) \in V(E)$ be such that $(\bar{z}_{i,j})(\bar{c}_{i,j}) \notin V(E)$. Let f be an element of q such that $f((\bar{z}_{i,j})(\bar{c}_{i,j})) \neq 0$. Let $\{\alpha_s\}$ be a basis of E considered as a vector space over C_k^Λ and let $f((\bar{z}_{i,j})(C_{i,j})) = \sum_{\alpha_s} \alpha_s f_{\alpha_s}((C_{i,j}))$ where the $C_{i,j}$ are indeterminates and the $f_{\alpha_s}((C_{i,j})) \in C_k^\Lambda\{C_{1,1}, \ldots, C_{n,n}\}_\Lambda$. By assumption (and the fact that linear independence over constants is preserved when one goes to extension fields), we have that there is an α_s such that $f_{\alpha_s}((\bar{c}_{i,j})) \neq 0$. Since C_k^Λ is a Π-differentially closed field, there must exist $(c_{i,j}) \in G(C_k^\Lambda)$ such that $f_{\alpha_s}(c_{i,j}) \neq 0$. This contradicts the fact that $f((\bar{z}_{i,j})(c_{i,j})) = 0$.

Therefore the map $(V \times_k G_k)(E) \to V(E)$ defined by $(z, g) \mapsto zg$ defines a morphism $V \times_k G_k \to V$. At the ring level, this morphism corresponds to a homomorphism of rings

$$\begin{aligned} k\left\{X_{i,j}, \tfrac{1}{\det}\right\}_\Pi/q &\to k\left\{X_{i,j}, \tfrac{1}{\det}\right\}_\Pi/q \otimes_{C_k^\Lambda} C_k^\Lambda\left[Y_{i,j}, \tfrac{1}{\det}\right]/I \\ &\simeq k\left\{X_{i,j}, \tfrac{1}{\det}\right\}_\Pi/q \otimes_k \left(k \otimes_{C_k^\Lambda} C_k^\Lambda\left\{Y_{i,j}, \tfrac{1}{\det}\right\}_\Pi/I\right) \end{aligned}$$

where the map is induced by $(X_{i,j}) \mapsto (r_{i,j})(Y_{i,j})$. We have to show that the morphism $f\colon V \times_k G_k \to V \times_k V$, given by $(z, g) \mapsto (zg, z)$ is an isomorphism of differential algebraic varieties over k. In terms of rings, we have to show that the k-algebra homomorphism $f^*\colon k\{V\} \bigotimes_k k\{V\} \to k\{V\} \otimes_{C_k^\Lambda} k\{G\}$ is an isomorphism. To do this it suffices to find a Π-field extension k' of k such that $1_{k'} \otimes_k f^*$ is an isomorphism. For this it suffices to find Π-field extension k' of k such that $V_{k'}$ is isomorphic to $G_{k'}$ as a $G_{k'}$-torsor over k' that is, for some field extension $k' \supset k$, the induced morphism of varieties over k', namely $V_{k'} \times_{k'} G_k' \to V_{k'}$, makes $V_{k'}$ into a trivial G-torsor over k'.

Let $k' = K$, the PPV-extension of k for the differential equation. We have already shown that $I = qK\{X_{i,j}, \frac{1}{\det}\}_\Pi \cap k_0\{Y_{i,j}, \frac{1}{\det}\}_\Pi$ and this fact implies that

$$\begin{aligned} K\{V\} &= K \otimes_k (k\{X_{i,j}, \tfrac{1}{\det}\}_\Pi/q) \\ &\cong K \otimes_{C_k^\Lambda} \left(C_k^\Lambda\{Y_{i,j}, \tfrac{1}{\det}\}_\Pi/I\right) = K \otimes_{C_k^\Lambda} C_k^\Lambda\{G\} = K\{G\} \end{aligned} \tag{9.4}$$

In other words, we found an isomorphism $h\colon V_K \cong G_K$. We still have to verify that V_K as a G torsor over K is, via h, isomorphic to the trivial torsor $G \times_{C_k^\Lambda} G_K \to G_K$. To do this it is enough to verify that the following diagram is commutative and we leave this to the reader. The coordinate ring $C_k^\Lambda\{G\}$ of the group appears in several places. To keep track of the variables, we will write $C_k^\Lambda\{G\}$ as $C_k^\Lambda\{T_{i,j}, \frac{1}{\det}\}_\Pi/\tilde{I}$ where $\tilde{I}$ is the ideal I with the variables $Y_{i,j}$ replaced by $T_{i,j}$.

$$\begin{array}{ccc} K \otimes_k k\{X_{i,j}, \frac{1}{\det}\}_\Pi/q & \longrightarrow & K\{X_{i,j}, \frac{1}{\det}\}_{\Pi'}/qK\{X_{i,j}, \frac{1}{\det}\}_\Pi \otimes_{C_k^\Lambda} C_k^\Lambda\{G\} \\ {\scriptstyle (X_{i,j})\mapsto(r_{i,j})(Y_{i,j})}\downarrow & & \downarrow{\scriptstyle (X_{i,j})\mapsto(r_{i,j})(Y_{i,j})} \\ K \otimes_{C_k^\Lambda} C_k^\Lambda\{Y_{i,j}, \frac{1}{\det}\}_\Pi/I & \to & K\{Y_{i,j}, \frac{1}{\det}\}_\Pi/(I)_\Pi \otimes_{C_k^\Lambda} C_k^\Lambda\{G\} \end{array}$$

□

In the above diagram, the top arrow represents the map $(X_{i,j}) \mapsto (X_{i,j})(T_{i,j})$ and the bottom arrow represents the map $(Y_{i,j}) \mapsto (Y_{i,j})(T_{i,j})$. Using this result (and its proof), we can now finish the proof of Theorem 9.5 by proving conclusion (4) of this theorem. As in the usual Picard–Vessiot theory, the proof depends on the following group theoretic facts. Let G be a linear differential algebraic group defined over a Π-differentially closed field C_k^Λ. For any $g \in G$ the map $\rho_g\colon G \to G$ given by $\rho_g(h) = hg$ is a differential polynomial isomorphism of G onto G and therefore corresponds to an isomorphism $\rho_g^*\colon C_k^\Lambda\{G\} \to C_k^\Lambda\{G\}$. In this way G acts on the ring $k\{G\}$. Let H be a normal linear differential algebraic subgroup of G. The following facts follow from results of [9] and [22]:

(1) The G-orbit $\{\rho_g^*(f) \mid g \in G(C_k^\Lambda)\}$ of any $f \in C_k^\Lambda\{G\}$ spans a finite dimensional C_k^Λ-vector space.

(2) The group G/H has the structure of a linear differential algebraic group (over C_k^Λ) and its coordinate ring $C_k^\Lambda\{G/H\}$ is isomorphic to the ring of H-invariants $C_k^\Lambda\{G\}^H$.

(3) The two rings $Qt(C_k^\Lambda\{G\})^H$ and $Qt(C_k^\Lambda\{G\}^H)$ are naturally Π-isomorphic, where $Qt(\,\cdot\,)$ denotes the total quotient ring.

We now can prove

Proposition 9.13. *Let K be a PPV-extension of k with Galois group G and let H be a normal Kolchin-closed subgroup. Then K^H is a PPV-extension of k.*

Proof. Let K be the quotient field of the PPV-ring $R = k\{Z, \frac{1}{\det}\}$. As we have already noted (*cf.*, (9.4)), we have

$$K \otimes_k R \cong K \otimes_{C_k^\Lambda} C_k^\Lambda\{G\}$$

that is, the torsor corresponding to R becomes trivial over K. The group G acts on $K \otimes_{C_k^\Lambda} C_k^\Lambda\{G\}$ by acting trivially on the left factor and via ρ^* on the right factor, or trivially on the left factor and with the Galois action on the right factor. In this way we have that $K \otimes_k R^H \cong K \otimes_{C_k^\Lambda} C_k^\Lambda\{G\}^H = K \otimes_{C_k^\Lambda} C_k^\Lambda\{G/H\}$ and that $K \otimes_k K^H \cong K \otimes_{C_k^\Lambda} Qt(k\{G\}^H)$ by the items enumerated above.

We now claim that R^H is finitely generated as a Π-ring over k, hence as a Δ-ring over k. Since $C_k^\Lambda\{G/H\}$ is a finitely generated Π-C_k^Λ-algebra, we have that there exist $f_1, \dots, f_s \in R^H$ that generate $K \otimes_k R^H$ as a Π-K-algebra. We claim that $f_1, \dots, f_s$ generate R^H as a Π-k-algebra. Let $\mathcal{M}$ be a k-basis of $k\{f_1, \dots, f_s\}_\Pi$. By assumption, any element of $f \in K \otimes_k R^H = K \otimes_k k\{f_1, \dots, f_s\}_\Pi$ can be written uniquely as $f = \sum_{u \in \mathcal{M}} a_u \otimes u$ where $a_u \in K$. The Galois group $G(C_k^\Lambda)$ of K over k also acts on $K \otimes_k R^H$ by acting as differential automorphisms of the left factor and trivially on the right factor. Write $1 \otimes f \in 1 \otimes R^H \subset K \otimes_k R^H$ as $1 \otimes f = \sum_{u \in \mathcal{M}} a_u \otimes u$ where $a_u \in K$. Applying $\sigma \in G(C_k^\Lambda)$ to $1 \otimes f$ we have $1 \otimes f = \sum_{u \in \mathcal{M}} \sigma(a_u) \otimes u$. Therefore $\sigma(a_u) = a_u$ for all $\sigma \in G(C_k^\Lambda)$. The parameterized Galois theory implies that $a_u \in k$ for all u. Therefore $f \in k\{f_1, \dots, f_s\}_\Pi$ and so $R^H = k\{f_1, \dots, f_s\}_\Pi$.

Using item (1) in the above list, we may assume that $f_1, \dots, f_s$ form a basis of a $G/H(C_k^\Lambda)$ invariant C_k^Λ-vector space. Let Θ be the free commutative semigroup generated by the elements of Λ. By Theorem 1, Chapter II of [21] (or Lemma D.11 of [40]), there exist $\theta_1 = 1, \dots, \theta_s \in \Theta$ such that

$$W = (\theta_i(f_j))_{1 \le i \le s, 1 \le j \le s}$$

is invertible. For each $\partial_i \in \Lambda$, we have that $A_i = (\partial_i W)W^{-1}$ is left invariant by the action of $G/H(k_0)$. Therefore each $A_i \in \mathrm{gl}_n(k)$. Furthermore, the A_i satisfy the integrability conditions. We have that K^H is generated as a Δ-field over k by the entries of W. Since the constants of K^H are C_k^Λ, we have that K^H is a PPV-field for the system $\partial_i Y = A_i Y$, $\partial_i \in \Lambda$. □

We can now complete the proof of conclusion (4) of Theorem 9.5. If $F = K^H$ is left invariant by $\mathrm{Gal}_\Delta(K/k)$ then restriction to F gives a homomorphism of $\mathrm{Gal}_\Delta(K/k)$ to $\mathrm{Gal}_\Delta(F/k)$. By the previous results, the kernel of this map is H so H is normal in $\mathrm{Gal}_\Delta(K/k)$. To show surjectivity we need to show that any $\phi \in \mathrm{Gal}_\Delta(F/k)$ extends to a $\tilde{\phi} \in \mathrm{Gal}_\Delta(K/k)$. This follows from the fact of the unicity of PPV-extensions.

Now assume that H is normal in $\mathrm{Gal}_\Delta(K/k)$ and that there exists an element $\tau \in \mathrm{Gal}_\Delta(K/k)$ such that $\tau(F) \neq F$. The Galois group of K over $\tau(F)$ is $\tau H \tau^{-1}$. Since $F \neq \tau(F)$ we have $H \neq \tau H \tau^{-1}$, a contradiction.

The last sentence of conclusion (4) follows from the above proposition.

9.5 Parameterized liouvillian extensions

In this section we will prove Theorem 3.12. One may recast this latter result in the more general setting of the last three sections but for simplicity we will stay with the original formulation. Let K and k be as in the hypotheses of this theorem. Let $K_A^{\mathrm{PV}} \subset K$ be the associated PV-extension as in Proposition 3.6.

(1) $\Rightarrow$ (2): Assume that the Galois group $\mathrm{Gal}_\Delta(K/k)$ contains a solvable subgroup of finite index. We may assume this subgroup is Kolchin closed. Since $\mathrm{Gal}_\Delta(K/k)$ is Zariski-dense in $\mathrm{Gal}_{\{\partial_0\}}(K_A^{\mathrm{PV}}/k)$, we have that this latter group also contains a solvable subgroup of finite index. Theorem 1.43 of [40] implies that K_A^{PV} is a liouvillian extension of k, that is, there is a tower of ∂_0-fields $k = K_0 \subset K_1 \subset \cdots \subset K_r = K_A^{\mathrm{PV}}$ such that $K_i = K_{i-1}(t_i)$ for $i = 1, \ldots, r$ where either $\partial_0 t_i \in K_{i-1}$, or $t_i \neq 0$ and $\partial_0 t_i / t_i \in K_{i-1}$ or t_i is algebraic over K_{i-1}. We can therefore form a tower of Δ-fields $k = \tilde{K}_0 \subset \tilde{K}_1 \subset \cdots \subset \tilde{K}_r$ by inductively defining $\tilde{K}_i = \tilde{K}_{i-1}\langle t_i \rangle_\Delta$. Since $K_A^{\mathrm{PV}} = K_r$, we have $K = \tilde{K}_r$ and so K is a parameterized liouvillian extension.

(3) $\Rightarrow$ (1): Assume that K is contained in a parameterized liouvillian extension of k. We wish to show that K_A^{PV} is contained in a liouvillian extension of k. For this we need the following lemma.

Lemma 9.14. *If L is a parameterized liouvillian extension of k then $L = \bigcup_{i \in \mathbb{N}} L_i$ where $L_{i+1} = L_i(\{t_{i,j}\}_{j \in \mathbb{N}})$ and $\{t_{i,j}\}$ is a set of elements such that for each j either $\partial_0 t_{i,j} \in L_i$ or $t_{i,j} \neq 0$ and $\partial_0 t_{i,j}/t_{i,j} \in L_i$ or $t_{i,j}$ is algebraic over L_i.*

Proof. In this proof we shall refer to a tower of fields $\{L_i\}$as above, as a ∂_0-tower for L. By induction on the length of the tower of Δ-fields defining L as a parameterized liouvillian extension of k, it is enough to show the following: *Let $\{L_i\}$ be a ∂_0-tower for the Δ-field L and let $L\langle t \rangle_\Delta$ be an extension of L such that $\partial_0 t \in L$, $\partial_0 t/t \in L$ or t is algebraic of L. Then there exists a ∂_0-liouvillian tower for $L\langle t \rangle_\Delta$.* We shall deal with three cases.

If t is algebraic over L, then it is algebraic over some L_{j-1}. We then inductively define $\tilde{L}_i = L_i$ if $i < j$, $\tilde{L}_j = L_j(t)$ and $\tilde{L}_i = L_i(\tilde{L}_j)$ if $i > j$. The fields $\{\tilde{L}_i\}$ are then a ∂_0-tower for $L\langle t \rangle_\Delta$.

Now, assume that $\partial_0 t = a \in L$. Let $\Theta = \{\partial_0^{n_0} \partial_1^{n_1} \ldots \partial_m^{n_m}\}$ be the commutative semigroup generated by the derivations of Δ. Note that $L\langle t \rangle_\Delta = L(\{\theta t\}_{\theta \in \Theta})$. For any $\theta \in \Theta$ we have $\partial_0(\theta t) = \theta(\partial_0 t) = \theta(a) \in L$. We define $\tilde{L}_i = L_i(\{\theta t \mid (\theta a) \in L_{i-1}\})$. Each $\tilde{L}_i$ contains $\tilde{L}_{i-1}$ and is an extension of $\tilde{L}$ of the correct type. Since $a \in L$, we have that for any $\theta \in \Theta$ there exists an i such that $\theta(a) \in L_{i-1}$, so $\theta(t) \in \tilde{L}_i$. Therefore, $\bigcup_{i \in \mathbb{N}} \tilde{L}_i = L\langle t \rangle_\Delta$ so $\{\tilde{L}\}$ is a ∂_0-tower for $L\langle t \rangle_\Delta$.

Finally assume that $\partial_0 t/t = a \in L_j \subset L$. For $\theta = \partial_0^{n_0} \partial_1^{n_1} \ldots \partial_m^{n_m} \in \Theta$, we define $\mathrm{ord}\theta = n_0 + n_1 + \cdots + n_m$. For any $\theta \in \Theta$, the Leibniz rule implies that $\theta(at) = p_\theta + a\theta t$ where

$$p_\theta \in \mathbb{Q}\big[\{\theta' a\}_{\mathrm{ord}(\theta') \leq \mathrm{ord}(\theta)}, \{\theta'' t\}_{\mathrm{ord}(\theta'') < \mathrm{ord}(\theta)}\big].$$

Note the strict inequality in the second subscript. Let $S_\theta = \{\theta' a\}_{\mathrm{ord}(\theta')\le \mathrm{ord}(\theta)} \cup \{\theta'' t\}_{\mathrm{ord}(\theta'')<\mathrm{ord}(\theta)}$. We define a new tower inductively:

$$\tilde{L}_1 = L_1(t), \quad \tilde{L}_i = \text{the compositum of } L_i \text{ and } \tilde{L}_{i-1}(\{\theta t \mid S_\theta \subset \tilde{L}_{i-1}\})$$

We now show that this is a ∂_0-tower for $L\langle t\rangle_\Delta$. We first claim that $\tilde{L}_i$ is an $\{\partial_0\}$-extension of $\tilde{L}_{i-1}$ generated by ∂_0-integrals or ∂_0-exponentials of integrals or elements algebraic over $\tilde{L}_{i-1}$. For $i = 1$, we have that $\partial_0 t/t \in L_0$ and L_1 is generated by such elements. For $i > 1$, assume $\theta \in \Theta$ and $S_\theta \subset \tilde{L}_{i-1}$. We then have that

$$\partial_0\left(\frac{\theta t}{t}\right) = \frac{p_\theta}{t} \in \tilde{L}_{t-i}$$

since $t, p_\theta \in \tilde{L}_{i-1}$. Therefore $\tilde{L}_{i-1}$ is generated by the correct type of elements.

We now show that for any $\theta \in \Theta$ there is some j such that $\theta(t) \in \tilde{L}_j$. We proceed by induction on $i = \mathrm{ord}(\theta)$. For $i = 0$ this is true by construction. Assume the statement is true for $\mathrm{ord}(\theta') < i$. Since there are only a finite number of such θ, there exists an $r \in \mathbb{N}$ such that $\{\theta'' t\}_{\mathrm{ord}(\theta'')<\mathrm{ord}(\theta)} \subset \tilde{L}_r$. Since $\{\theta' a\}_{\mathrm{ord}(\theta')\le\mathrm{ord}(\theta)}$ is a finite subset of L, there is an $s \in \mathbb{N}$ such that $\{\theta' a\}_{\mathrm{ord}(\theta')\le\mathrm{ord}(\theta)} \subset L_s$. Therefore for $j > \max(r, s)$, $\theta t \in \tilde{L}_j$. Thus, $\bigcup_{i\in\mathbb{N}} \tilde{L}_i = L\langle t\rangle_\Delta$ so $\{\tilde{L}\}$ is a ∂_0-tower for $L\langle t\rangle_\Delta$. □

Let L be a parameterized liouvillian extension of k containing the field K. Lemma 9.14 implies that K_A^{PV} lies in a ∂_0-tower. Since K_A^{PV} is finitely generated, one sees that this implies that K_A^{PV} lies in a liouvillian extension of k. Therefore the PV-group $\mathrm{Gal}_\Delta(K_A^{\mathrm{PV}}/k)$ has a solvable subgroup H of finite index. Since we can identify $\mathrm{Gal}_{\{\partial_0\}}(K/k)$ with a subgroup of $\mathrm{Gal}_\Delta(K_A^{\mathrm{PV}}/k)$, we have that $\mathrm{Gal}_\Delta(K/k) \cap H$ is a solvable subgroup of finite index in $\mathrm{Gal}_\Delta(K/k)$. □

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On the reductions and classical solutions of the Schlesinger equations

*Boris Dubrovin and Marta Mazzocco**

SISSA, International School of Advanced Studies
via Beirut 2-4, 34014 Trieste, Italy
email: `dubrovin@sissa.it`

School of Mathematics
The University of Manchester
Manchester M60 1QD, United Kingdom
email: `Marta.Mazzocco@manchester.ac.uk`

To the memory of our friend Andrei Bolibruch

Abstract. The Schlesinger equations $S_{(n,m)}$ describe monodromy preserving deformations of order m Fuchsian systems with $n+1$ poles. They can be considered as a family of commuting time-dependent Hamiltonian systems on the direct product of n copies of $m \times m$ matrix algebras equipped with the standard linear Poisson bracket. In this paper we address the problem of reduction of particular solutions of "more complicated" Schlesinger equations $S_{(n,m)}$ to "simpler" $S_{(n',m')}$ having $n' \leq n$, $m' \leq m$.

Contents

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1 Introduction

The *Schlesinger equations* $S_{(n,m)}$ [35] is the following system of nonlinear differential equations

$$\frac{\partial}{\partial u_j} A_i = \frac{[A_i, A_j]}{u_i - u_j}, \quad i \neq j, \qquad \frac{\partial}{\partial u_i} A_i = -\sum_{j \neq i} \frac{[A_i, A_j]}{u_i - u_j}, \tag{1.1}$$

for $m \times m$ matrix valued functions $A_1 = A_1(u), \ldots, A_n = A_n(u)$, where the independent variables $u = (u_1, \ldots, u_n)$ must be pairwise distinct. The first non-trivial case $S_{(3,2)}$ of the Schlesinger equations corresponds to the famous sixth Painlevé equation [9], [35], [10], the most general of all Painlevé equations. In the case of any number $n > 3$ of 2×2 matrices A_j, the Schlesinger equations reduce to the Garnier systems $\mathcal{G}_n$ (see [10], [11], [32]).

The Schlesinger equations $S_{(n,m)}$ appeared in the theory of *isomonodromic deformations* of Fuchsian systems. Namely, the monodromy matrices of the Fuchsian system

$$\frac{d\Phi}{dz} = \sum_{k=1}^{n} \frac{A_k(u)}{z - u_k} \Phi, \quad z \in \mathbb{C} \setminus \{u_1, \ldots, u_n\} \tag{1.2}$$

do not depend on $u = (u_1, \ldots, u_n)$ if the matrices $A_i(u)$ satisfy (1.1). Conversely, under certain assumptions on the matrices $A_1, \ldots, A_n$ and for the matrix

$$A_\infty := -(A_1 + \cdots + A_n), \tag{1.3}$$

all isomonodromic deformations of the Fuchsian system are given by solutions to the Schlesinger equations (see, e.g., [36])[1].

The solutions to the Schlesinger equations can be parameterized by the *monodromy data* of the Fuchsian system (1.2) (see precise definition below in Section 2). To reconstruct the solution starting from given monodromy data one is to solve the classical *Riemann–Hilbert problem*. The main outcome of this approach says that the solutions $A_i(u)$ can be continued analytically to meromorphic functions on the universal covering of

$$\left\{(u_1, \ldots, u_n) \in \mathbb{C}^n \mid u_i \neq u_j \text{ for } i \neq j\right\}$$

[24], [30]. This is a generalization of the celebrated *Painlevé property* of absence of movable critical singularities (see details in [14], [15]). In certain cases the technique based on the theory of Riemann–Hilbert problem gives a possibility to compute the asymptotic behavior of the solutions to the Schlesinger equations near the critical

[1] Bolibruch constructed non-Schlesinger isomonodromic deformations in [4]. These can occur when the matrices A_i are resonant, i.e. admit pairs of eigenvalues with positive integer differences. To avoid such non-Schlesinger isomonodromic deformations, we need to extend the set of monodromy data (see Section 2 below).

locus $u_i = u_j$ for some $i \neq j$, although, in general, the problem of determining the asymptotic behaviour near the critical points is still open [17], [7], [12], [5].

It is the Painlevé property that was used by Painlevé and Gambier as the basis for their classification scheme of nonlinear differential equations. Of the list of some 50 second order nonlinear differential equations possessing Painlevé property the six (nowadays known as *Painlevé equations*) are selected due to the following crucial property: the general solutions to these six equations cannot be expressed in terms of *classical functions*, i.e., elementary functions, elliptic and other classical transcendental functions (see [38] for a modern approach to this theory based on a nonlinear version of the differential Galois theory). In particular, according to these results the general solution to the Schlesinger system $S_{(3,2)}$ corresponding to Painlevé-VI equation cannot be expressed in terms of classical functions.

A closely related question is the problem of construction and classification of *classical solutions* to Painlevé equations and their generalizations. This problem remains open even for the case of Painlevé-VI although there are interesting results, based on the theory of symmetries of Painlevé equations [34], [33], [1] and on the geometric approach to studying the space of monodromy data [7], [13], [27], [28].

The above methods do not give any clue to solution of the following general problems: are solutions of $S_{(n+1,m)}$ or of $S_{(n,m+1)}$ more complicated than those of $S_{(n,m)}$? Which solutions to $S_{(n+1,m)}$ or $S_{(n,m+1)}$ can be expressed via solutions to $S_{(n,m)}$? Furthermore, which of them can ultimately be expressed via classical functions?

In this paper we aim to suggest a general approach to the theory of reductions and classical solutions of the general Schlesinger equations $S_{(n,m)}$ for all n, m, based on the Riemann–Hilbert problem and on the group-theoretic properties of the monodromy group of the linear system (1.2). Our approach consists in determining the monodromy data of the Fuchsian system (1.2) that guarantee to have a reduction to $S_{(n-1,m)}$ or $S_{(n,m-1)}$ and eventually a classical solution.

We need a few definitions. Let us fix a solution to the Schlesinger equations $S_{(n,m)}$. Applying the algebraic operations and differentiations to the matrix entries of this solution we obtain a field $\mathcal{S}_{(n,m)}$ equipped with n pairwise commuting differentiations $\partial/\partial u_1, \ldots, \partial/\partial u_n$, to be short a *differential field*. Define the *rational closure* $\mathcal{K}$ of a differential field $\mathcal{S}$ represented by functions of n variables by taking all rational functions with coefficients in $\mathcal{S}$

$$\mathcal{K} := \mathcal{S}(u_1, \ldots, u_n).$$

Taking the rational closure of the differential field $S_{(n,m)}$, we obtain the differential field $\mathcal{K}_{(n,m)}$. (Needless to say that the field $\mathcal{K}_{(n,m)}$ depends on the choice of the solution to the Schlesinger equations $S_{(n,m)}$.)

We now construct new differential fields obtained from $\mathcal{K}_{(n_1,m_1)}, \ldots, \mathcal{K}_{(n_k,m_k)}$ by applying one or more of the following *admissible* elementary operations:

1. *Tensor product*. Given two differential fields $\mathcal{K}_1$ and $\mathcal{K}_2$ represented by functions of n_1 and n_2 variables $u_1, \ldots, u_{n_1}$ and $v_1, \ldots, v_{n_2}$ respectively, produce a new differential field $\mathcal{K}_1 \otimes \mathcal{K}_2$ taking the rational closure of the minimal differential field

of functions of n_1+n_2 independent variables $u_1, \dots, u_{n_1}, v_1, \dots, v_{n_2}$ containing both K_1 and K_2. A particular case of this operation is

2. *Addition of an independent variable*. Given a differential field $\mathcal{K}$ represented by functions of n variables $u_1, \dots, u_n$ define an extension $\widetilde{\mathcal{K}} \supset \mathcal{K}$ by taking rational functions of a new independent variable u_{n+1} with coefficients in $\mathcal{K}$,

$$\widetilde{\mathcal{K}} = \mathcal{K} \otimes \mathbb{C}(u_{n+1}).$$

3. Given two differential fields $\mathcal{K}_1$, $\mathcal{K}_2$ represented by functions of the same number of variables n, define the *composite* $\mathcal{K}_1\mathcal{K}_2$ taking the minimal differential field of functions of n variables containing both $\mathcal{K}_1$ and $\mathcal{K}_2$ and applying the rational closure procedure.

4. A differential field extension $\mathcal{K}' \supset \mathcal{K}$ is said to be of the *Picard–Vessiot type* if it is the minimal rationally closed differential field of functions of n variables containing solutions of a Pfaffian linear system with coefficients in $\mathcal{K}$ [20], [37].

Recall that a Pfaffian linear system of order k with coefficients in a differential field $\mathcal{K}$ represented by functions of n variables reads

$$\frac{\partial Y}{\partial u_i} = B_i Y, \quad i = 1, \dots, n, \quad Y = (y_1, \dots, y_k)^T$$

where the matrices $B_i \in \mathrm{Mat}(k; \mathcal{K})$ must satisfy

$$\frac{\partial B_i}{\partial u_j} - \frac{\partial B_j}{\partial u_i} + [B_i, B_j] = 0 \quad \text{for all } i \neq j.$$

The linear space of solutions of the Pfaffian system is finite dimensional. The differential field $\mathcal{K}'$ is the minimal extension of $\mathcal{K}$ containing all components $y_1, \dots, y_k$ of any of these solutions.

We will also denote $\mathcal{K}^{(N)}$ the differential field extension of $\mathcal{K}$ obtained by N Picard–Vessiot type extensions of $\mathcal{K}$:

$$\mathcal{K} \subset \mathcal{K}' \subset \mathcal{K}'' \subset \cdots \subset \mathcal{K}^{(N)}.$$

Using the above admissible extensions we can describe in what circumstances a particular solution to the Schlesinger equations $S_{(n,m)}$ can be expressed via solutions to $S_{(n',m')}$ with smaller n' or m'. Similar results were obtained in [29] for the special case of $m = 2$.

Theorem 1.1. *Consider a solution to $S_{(n,m)}$ such that the eigenvalues of the matrix A_∞ are pairwise distinct and the monodromy group of the associated Fuchsian system* (1.2) *admits a k-dimensional invariant subspace, $k > 0$. Then this solution belongs to a Picard–Vessiot type extension $\mathcal{K}^{(N)}$ for some N of the composite*

$$\mathcal{K} = \mathcal{K}_{(n,k)}\mathcal{K}_{(n,m-k)}$$

where $\mathcal{K}_{(n,k)}$ and $\mathcal{K}_{(n,m-k)}$ are two differential fields associated with certain two solutions of the Schlesinger equations $S_{(n,k)}$ and $S_{(n,m-k)}$ respectively.

In particular,

Corollary 1.2. *Given a solution to* $S_{(n,m)}$ *such that the monodromy group of the associated Fuchsian system* (1.2) *is upper-triangular and the eigenvalues of* A_∞ *are pairwise distinct, it belongs to a Picard–Vessiot type extension* $\mathcal{K}_0^{(N)}$ *for some* N *of*

$$\mathcal{K}_0 = \mathbb{C}(u_1, \dots, u_n).$$

The proof of this theorem is based on the following two lemmata.

Lemma 1.3. *Given a solution*

$$A(z; u) = \sum_{i=1}^{n} \frac{A_i(u)}{z - u_i} \tag{1.4}$$

to the Schlesinger equations $S_{(n,m)}$ *with diagonalizable matrix* A_∞ *such that the associated monodromy representation has a* k*-dimensional invariant subspace, denote* $\mathcal{K}_{(n,m)}$ *the corresponding differential field. Then there exists a matrix*

$$G(z; u) \in \overline{\mathcal{K}}_{(n,m)}(z), \quad \det G(z) \equiv 1$$

such that all matrices $B_i(u)$, $i = 1, \dots, n$ *of the gauge equivalent Fuchsian system with*

$$B(z; u) = G^{-1}(z; u) A(z; u) G(z; u) + G^{-1}(z; u) \frac{dG(z; u)}{dz} = \sum_{i=1}^{n} \frac{B_i(u)}{z - u_i} \tag{1.5}$$

have a u*-independent* k*-dimensional common invariant subspace. Here* $\overline{\mathcal{K}}_{(n,m)}$ *is a Picard–Vessiot type extension of the field* $\mathcal{K}_{(n,m)}$*. Moreover, the matrices* $B_1(u), \dots, B_n(u)$ *satisfy the Schlesinger equations.*

This lemma, apart from polynomiality of the gauge transformation in z, is the main result of the papers [22], [23] by S. Malek[2]. We give here a new short proof of this result (for the sake of technical simplicity we add the assumption of diagonalizability of the matrix A_∞) by presenting a reduction algorithm consisting of a number of elementary and explicitly written transformations.

It is a one-line calculation that shows that the $S_{(n,m)}$ Schlesinger equations for the matrices $B_1(u), \dots, B_n(u)$ of the form

$$B_i(u) = \begin{pmatrix} B_i'(u) & C_i(u) \\ 0 & B_i''(u) \end{pmatrix},$$

where $B_i'(u)$ and $B_i''(u)$ are respectively $k \times k$ and $(m-k) \times (m-k)$ matrices, reduces to the $S_{(n,k)}$ and $S_{(n,m-k)}$ Schlesinger systems for the matrices $B_i'(u)$ and $B_i''(u)$ and

[2] Actually, there is a stronger claim in the main result of [23], namely, it is said that the coefficients of the reducing gauge transformation are rational functions in $u_1, \dots, u_n$ and entries of $A_1(u), \dots, A_n(u)$. We were unable to reproduce this result.

to the linear Pfaffian equations

$$\partial_j C_i = \frac{1}{u_i - u_j}\big(B_i' C_j - B_j' C_i + C_i B_j'' - C_j B_i''\big), \quad j \neq i,$$
$$\partial_i C_i = -\sum_{j\neq i} \frac{1}{u_i - u_j}\big(B_i' C_j - B_j' C_i + C_i B_j'' - C_j B_i''\big).$$

Therefore the Schlesinger deformation of the reduced system (1.5) belongs to a Picard–Vessiot type extension of the composite $\mathcal{K}_{n,k}\mathcal{K}_{n,m-k}$.

To complete the proof of Theorem 1.1 we need to invert the above gauge transformation, i.e., to express the coefficients of the original Fuchsian system (1.4) via the solution of the reduced system (1.5).

Lemma 1.4. (i) *For a Fuchsian system* (1.4) *satisfying the assumptions of the previous lemma, the monodromy data, in the sense of Definition* 2.5 *here below,*

$$\Lambda^{(1)}(A), R^{(1)}(A), \dots, \Lambda^{(\infty)}(A), R^{(\infty)}(A), C_1(A), \dots, C_n(A)$$

of the system (1.4) *and*

$$\Lambda^{(1)}(B), R^{(1)}(B), \dots, \Lambda^{(\infty)}(B), R^{(\infty)}(B), C_1(B), \dots, C_n(B)$$

of (1.5) *are related by*

$$\begin{aligned} \Lambda^{(i)}(B) &= P^{-1}\Lambda^{(i)}(A)P, \quad i = 1, \dots, n,\\ R^{(i)}(B) &= P^{-1}R^{(i)}(A)P, \quad i = 1, \dots, \infty,\\ C^{(i)}(B) &= P^{-1}C^{(i)}(A)P, \quad i = 1, \dots, n,\\ \Lambda^{(\infty)}(B) &= P^{-1}\Lambda^{(\infty)}(A)P + \mathrm{diag}(N_1, \dots, N_m), \quad N_i \in \mathbb{Z}. \end{aligned} \tag{1.6}$$

Here $P \in S_m$ is a permutation matrix.

(ii) *Denote $\mathcal{K}^A_{n,m}$ and $\mathcal{K}^B_{n,m}$ the differential fields associated with the Schlesinger deformations of two systems* (1.4) *and* (1.5) *respectively. If the monodromy data of the systems are related as in* (1.6) *then there exists a matrix*

$$\widetilde{G}(z; u) \in \overline{\mathcal{K}}^B_{n,m}[z], \quad \det \widetilde{G}(z; u) \equiv 1$$

such that

$$A(z) \equiv \widetilde{G}^{-1}(z; u)B(z; u)\widetilde{G}(z; u) + \widetilde{G}^{-1}(z; u)\frac{d\widetilde{G}(z; u)}{dz}.$$

Here, like in Lemma 1.3*, $\overline{\mathcal{K}}^B_{n,m}$ is a suitable Picard–Vessiot type extension of the field $\mathcal{K}^B_{n,m}$.*

We obtain therefore two inclusions

$$\mathcal{K}^B_{n,m} \subset \overline{\mathcal{K}}^A_{n,m}, \quad \mathcal{K}^A_{n,m} \subset \overline{\mathcal{K}}^B_{n,m}. \tag{1.7}$$

Theorem 1.1 easily follows from the above statements.

Let us proceed now to the second mechanism of reducing the Schlesinger equations. Let us assume that l monodromy matrices $M_{i_1}, \dots, M_{i_l}$ of the Fuchsian system of the form (1.2), are *scalar matrices*. In that case we will call the solution $A_1(u), \dots, A_n(u)$ is *l-smaller*. We call *l-erased* the Fuchsian system $S_{n-l,m}$ of the same size with the poles $z = u_{i_1}, \dots, z = u_{i_l}$ erased.

Theorem 1.5. *Let $A_1, \dots, A_n$ be an l-smaller solution of the Schlesinger equations. Then $A_1(u), \dots, A_n(u)$ belong to the differential field obtained by admissible extensions from $\mathcal{K}_{(n-l,m)}$, the rational closure of the differential field $\mathcal{S}_{n-l,m}$ associated with a solution to the l-erased Fuchsian system $S_{(n-l,m)}$. In particular, if $l = n-2$ then $A_1, \dots, A_n$ belong to the differential field obtained by admissible extensions from $\mathbb{C}(u_1, \dots, u_n)$.*

The proof of this theorem consists first in observing that, due to the fact that all matrices $A_1, \dots, A_n$ can be assumed to be traceless, any scalar matrix M_k must have the form

$$M_k = e^{\frac{2\pi i p}{m}} \mathbb{1}, \quad p \in \mathbb{Z}.$$

As a first step we assume M_k, say for $k = n$, to be the identity and we construct a gauge transformation in a suitable Picard–Vessiot type extension of the field $\mathcal{K}^A_{n,m}$ defined by the solution $A_1, \dots, A_n$, which maps A_n to zero without changing the nature of the other singular points $u_1, \dots, u_{n-1}$, nor introducing new ones. In this way we obtain a new solution $B_1, \dots, B_{n-1}$ of the Schlesinger equations $S_{n-1,m}$. We then prove that the original solution $A_1, \dots, A_n$ can be constructed in terms of $B_1, \dots, B_{n-1}$ by means of admissible operations.

When $M_k = e^{\frac{2\pi i p}{m}} \mathbb{1}$ is not the identity, we need to map $A_1, \dots, A_n$ bi-rationally to a new solution $\tilde{A}_1, \dots, \tilde{A}_n$ of the Schlesinger equations with $\widetilde{M}_k = \mathbb{1}$. To this end we apply the birational canonical transformations of Schlesinger equations found in [8][3].

To present here this class of transformations let us briefly remind the canonical Hamiltonian formulation of Schlesinger equations $S_{(n,m)}$ of [8].

Recall [19], [25] that Schlesinger equations can be written as Hamiltonian systems on the Lie algebra

$$\mathfrak{g} := \bigoplus_{i=1}^{n} \mathfrak{gl}(m) \ni (A_1, \dots, A_n)$$

with respect to the standard linear Lie–Poisson bracket on $\mathfrak{g}^*$ with some quadratic time-dependent Hamiltonians of the form

$$H_k := \sum_{l \neq k} \frac{\operatorname{tr}(A_k A_l)}{u_k - u_l}. \tag{1.8}$$

[3] An alternative way, as it was proposed by the referee, would be to replace our canonical transformations by a combination of Schlesinger transformations of [18] with scalar shifts instead. However, the birationality of the proposed transformation needs to be justified in the resonant case.

Because of isomonodromicity they can be restricted onto the symplectic leaves

$$\mathcal{O}_1 \times \cdots \times \mathcal{O}_n \in \mathfrak{g}^*$$

obtained by fixation of the conjugacy classes $\mathcal{O}_1, \dots, \mathcal{O}_n$ of the matrices $A_1, \dots, A_n$. The matrix A_∞ given in (1.3) is a common integral of the Schlesinger equations. Applying the procedure of symplectic reduction [26] we obtain the reduced symplectic space

$$\begin{gathered}\{A_1 \in \mathcal{O}_1, \dots, A_n \in \mathcal{O}_n,\ A_\infty = \text{given diagonal matrix}\}\\ \text{modulo simultaneous diagonal conjugations.}\end{gathered} \tag{1.9}$$

The dimension of this reduced symplectic leaf in the generic situation is equal to $2g$ where

$$g = \frac{m(m-1)(n-1)}{2} - (m-1).$$

In [8] a new system of the so-called *isomonodromic Darboux coordinates* $q_1, \dots, q_g$, $p_1, \dots, p_g$ on generic symplectic manifolds (1.9) was constructed and the Hamiltonians were expressed in these coordinates. Let us explain this construction.

The Fuchsian system (1.2) can be reduced to a scalar differential equation of the form

$$y^{(m)} = \sum_{l=0}^{m-1} d_l(z) y^{(l)}. \tag{1.10}$$

For example, one can eliminate last $m-1$ components of the vector function Φ to obtain an m-th order equation for the first component $y := \Phi_1$. (Observe that the reduction procedure depends on the choice of the component of Φ.) The resulting Fuchsian equation will have regular singularities at the same points $z = u_1, \dots, z = u_n, z = \infty$. It will also have other singularities produced by the reduction procedure. However, they will be *apparent* singularities, i.e., the solutions to (1.10) will be analytic in these points. Generically there will be exactly g apparent singularities (cf. [31]; a more precise result about the number of apparent singularities working also in the nongeneric situation was obtained in [3]); they are the first part $q_1, \dots, q_g$ of the canonical coordinates. The conjugated momenta are defined by

$$p_i = \operatorname{Res}_{z=q_i}\left(d_{m-2}(z) + \tfrac{1}{2} d_{m-1}^2(z)\right), \quad i = 1, \dots, g.$$

Theorem 1.6 ([8]). *Let the eigenvalues of the matrices $A_1, \dots, A_n, A_\infty$ be pairwise distinct. Then the map*

$$\left\{\begin{matrix}\textit{Fuchsian systems with given poles,}\\ \textit{given eigenvalues of } A_1, \dots, A_n, A_\infty\\ \textit{modulo diagonal conjugations}\end{matrix}\right\} \to (q_1, \dots, q_g, p_1, \dots, p_g) \tag{1.11}$$

gives a system of rational Darboux coordinates on the generic reduced symplectic leaf (1.9). *The Schlesinger equations $S_{(n,m)}$ in these coordinates are written in the* *canonical Hamiltonian form*

$$\frac{\partial q_i}{\partial u_k} = \frac{\partial \mathcal{H}_k}{\partial p_i} \quad \frac{\partial p_i}{\partial u_k} = -\frac{\partial \mathcal{H}_k}{\partial q_i}$$

with the Hamiltonians

$$\mathcal{H}_k = \mathcal{H}_k(q, p; u) = -\mathrm{Res}_{z=u_k}\big(d_{m-2}(z) + \tfrac{1}{2}d_{m-1}^2(z)\big), \quad k = 1, \dots, n.$$

Here *rational Darboux coordinates* means that the elementary symmetric functions $\sigma_1(q), \dots, \sigma_g(q)$ and $\sigma_1(p), \dots, \sigma_g(p)$ are rational functions of the coefficients of the system and of the poles $u_1, \dots, u_n$. Moreover, there exists a section of the map (1.11) given by rational functions

$$A_i = A_i(q, p), \quad i = 1, \dots, n, \tag{1.12}$$

symmetric in $(q_1, p_1), \dots, (q_g, p_g)$ with coefficients depending on $u_1, \dots, u_n$ and on the eigenvalues if the matrices A_i, $i = 1, \dots, n, \infty$. All other Fuchsian systems with the same poles $u_1, \dots, u_n$, the same eigenvalues and the same $(p_1, \dots, p_g, q_1, \dots, q_g)$ are obtained by simultaneous diagonal conjugation

$$A_i(q, p) \mapsto C^{-1} A_i(q, p) C, \quad i = 1, \dots, n, \quad C = \mathrm{diag}(c_1, \dots, c_m).$$

Theorem 1.7 ([8]). *The Schlesinger equations $S_{(n,m)}$ written in the canonical form of Theorem* 1.6 *admit a group of birational canonical transformations $\langle S_2, \dots, S_m, S_\infty\rangle$*

$$S_k: \begin{cases} \tilde{q}_i = u_1 + u_k - q_i, & i = 1, \dots, g, \\ \tilde{p}_i = -p_i, & i = 1, \dots, g, \\ \tilde{u}_l = u_1 + u_k - u_l, & l = 1, \dots, n, \\ \widetilde{\mathcal{H}}_l = -\mathcal{H}_l, & l = 1, \dots, n, \end{cases} \tag{1.13}$$

$$S_\infty: \begin{cases} \tilde{q}_i = \frac{1}{q_i - u_1}, & i = 1, \dots, g, \\ \tilde{p}_i = -p_i q_i^2 - \frac{2m^2-1}{m} q_i, & i = 1, \dots, g, \\ \tilde{u}_l = \frac{1}{u_l - u_1}, & l = 2, \dots, n, \\ u_1 \mapsto \infty, & \\ \infty \mapsto u_1, & \\ \widetilde{H}_1 = H_1, & \\ \widetilde{H}_l = -H_l(u_l - u_1)^2 + (u_l - u_1)(d_{m-1}^0(u_l - u_1))^2 & \\ \qquad - (u_l - u_1)\dfrac{(m-1)(m^2-m-1)}{m} d_{m-1}^0(u_l - u_1), & l = 2, \dots, n \end{cases} \tag{1.14}$$

where $d^0_{m-1}(u_k) = \sum_{s=1}^g \frac{1}{u_k-q_s} - \frac{m\,(m-1)}{2}\sum_{l\neq k}\frac{1}{u_k-u_l}$. *The transformation* S_k *acts on the monodromy matrices as follows*

$$\begin{aligned}\widetilde{M}_1 &= M_1^{-1}\dots M_{k-1}^{-1}M_kM_{k-1}\dots M_1,\\ \widetilde{M}_j &= M_{j-1}, \quad j=2,\dots,k,\\ \widetilde{M}_i &= M_i, \qquad i=k+1,\dots,n.\end{aligned}$$

The transformation S_∞ *acts on the monodromy matrices as follows:*

$$\widetilde{M}_\infty = e^{-\frac{2\pi i}{m}}M_1, \quad \widetilde{M}_1 = e^{\frac{2\pi i}{m}}M_\infty, \quad \widetilde{M}_j = M_1^{-1}M_jM_1 \quad \textit{for } j=2,\dots,n.$$

To conclude, Theorems 1.1 and 1.5 show that for certain very special monodromy groups the Schlesinger equations $S_{(n,m)}$ *reduce* to solutions of $S_{(n',m')}$ with $n' < n$ and/or $m' < m$. We do not know any other general mechanism of reducibility of Schlesinger equations[4]. As generically the monodromy group of the system (1.2) is not reducible nor smaller, we expect that generic solutions of the Schlesinger equations $S_{(n,m)}$ do not belong to any admissible extension of composites of the differential fields of the form $\mathcal{K}_{(n',m')}$ with $n' < n$ and/or $m' < m$. The proof of this fact, that is the proof of *irreducibility* of the Schlesinger equations, is still a rather intriguing open problem.

2 Schlesinger equations as monodromy preserving deformations of Fuchsian systems

In this section we establish our notations, remind a few basic definitions and prove some technical lemmata that will be useful throughout this paper.

The Schlesinger equations $S_{(n,m)}$ describe monodromy preserving deformations of Fuchsian systems (1.2) with $n+1$ regular singularities at $u_1,\dots,u_n,u_{n+1}=\infty$:

$$\frac{\mathrm{d}}{\mathrm{d}z}\Phi = \sum_{k=1}^n \frac{A_k}{z-u_k}\Phi, \quad z\in\mathbb{C}\setminus\{u_1,\dots,u_n\}. \tag{2.1}$$

A_k being $m\times m$ matrices independent of z, and $u_k\neq u_l$ for $k\neq l, k,l=1,\dots,n+1$. Let us explain the precise meaning of this claim.

[4]A different mechanism suggested in [7] for producing *algebraic* solutions to the Schlesinger equations by studying the finite orbits of the natural action of the braid group B_n on the representation variety $\mathrm{Hom}(\pi_1(\mathbb{C}\backslash\{u_1,\dots,u_n\})\to \mathrm{SL}(m,\mathbb{C}))$ will not be discussed in the present work.

2.1 Levelt basis near a logarithmic singularity and local monodromy data

A system

$$\frac{d\Phi}{dz} = \frac{A(z)}{z - z_0}\Phi \tag{2.2}$$

is said to have a *logarithmic*, or *Fuchsian* singularity at $z = z_0$ if the $m \times m$ matrix valued function $A(z)$ is analytic in some neighborhood of $z = z_0$. By definition the *local monodromy data* of the system is the class of equivalence of such systems w.r.t. local gauge transformations

$$A(z) \mapsto G^{-1}(z)A(z)\, G(z) + (z - z_0)G^{-1}(z)\partial_z G(z) \tag{2.3}$$

analytic near $z = z_0$ satisfying

$$\det G(z_0) \neq 0.$$

The local monodromy can be obtained by choosing a suitable fundamental matrix solution of the system (2.2). The most general construction of such a fundamental matrix was given by Levelt [21]. We will briefly recall this construction in the form suggested in [6].

Without loss of generality one can assume that $z_0 = 0$. Expanding the system near $z = 0$ one obtains

$$\frac{d\Phi}{dz} = \left(\frac{A_0}{z} + A_1 + z\, A_2 + \cdots\right)\Phi. \tag{2.4}$$

Let us now describe the structure of local monodromy data.

Two linear operators Λ, R acting in the complex m-dimensional space V

$$\Lambda,\ R\colon V \to V$$

are said to form an *admissible pair* if the following conditions are fulfilled.
1. The operator Λ is semisimple and the operator R is nilpotent.
2. R commutes with $e^{2\pi i \Lambda}$,

$$e^{2\pi i \Lambda} R = R\, e^{2\pi i \Lambda}. \tag{2.5}$$

Observe that, due to the last condition the operator R satisfies

$$R(V_\lambda) \subset \bigoplus_{k\in\mathbb{Z}} V_{\lambda+k} \quad \text{for any } \lambda \in \operatorname{Spec}\Lambda, \tag{2.6}$$

where $V_\lambda \subset V$ is the subspace of all eigenvectors of Λ with the eigenvalue λ. The last condition says that
3. The sum in the r.h.s. of (2.6) contains only non-negative values of k.

A decomposition

$$R = R_0 + R_1 + R_2 + \cdots \tag{2.7}$$

is defined where

$$R_k(V_\lambda) \subset V_{\lambda+k} \quad \text{for any } \lambda \in \operatorname{Spec}\Lambda. \tag{2.8}$$

Clearly this decomposition contains only a finite number of terms. Observe the useful identity

$$z^{\Lambda} R\, z^{-\Lambda} = R_0 + z\, R_1 + z^2 R_2 + \cdots . \tag{2.9}$$

Theorem 2.1. *For a system* (2.4) *with a logarithmic singularity at* $z = 0$ *there exists a fundamental matrix solution of the form*

$$\Phi(z) = \Psi(z) z^{\Lambda} z^{R} \tag{2.10}$$

where $\Psi(z)$ *is a matrix valued function analytic near* $z = 0$ *satisfying*

$$\det \Psi(0) \neq 0$$

and Λ, R *is an admissible pair.*

The formula (2.10) makes sense after fixing a branch of logarithm $\log z$ near $z = 0$. Note that z^R is a polynomial in $\log z$ due to nilpotency of R.

The proof can be found in [21] (cf. [6]). Clearly Λ is the semisimple part of the matrix A_0; R_0 coincides with its nilpotent part. The remaining terms of the expansion appear only in the *resonant case*, i.e., if the difference between some eigenvalues of Λ is a positive integer. In the important particular case of a diagonalizable matrix A_0,

$$T^{-1} A_0 T = \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$$

with some nondegenerate matrix T, the matrix function $\Psi(z)$ in the fundamental matrix solution (2.10) can be obtained in the form

$$\Psi(z) = T\left(\mathbb{1} + z\, \Psi_1 + z^2 \Psi_2 + \cdots\right).$$

The matrix coefficients Ψ_1, $\Psi_2, \ldots$ of the expansion as well as the components R_1, $R_2, \ldots$ of the matrix R (see (2.7)) can be found recursively from the equations

$$[\Lambda, \Psi_k] - k\, \Psi_k = -B_k + R_k + \sum_{i=1}^{k-1} \Psi_{k-i} R_i - B_i \Psi_{k-i}, \quad k \geq 1.$$

Here $B_k := T^{-1} A_k T$, $k \geq 1$. If $k_{\max}$ is the maximal integer among the differences $\lambda_i - \lambda_j$ then

$$R_k = 0 \quad \text{for } k > k_{\max}.$$

Observe that vanishing of the logarithmic terms in the fundamental matrix solution (2.10) is a constraint imposed only on the first $k_{\max}$ coefficients $A_1, \ldots, A_{k_{\max}}$ of the expansion (2.4).

Example 2.2. For the Fuchsian system (1.2) having diagonal the matrix

$$A_\infty = -(A_1 + \cdots + A_n) = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$$

the fundamental matrix of Theorem 2.1 has the form

$$\Phi = \left(1 + \frac{\Psi_1}{z} + \mathcal{O}\left(1/z^2\right)\right) z^{-\Lambda} z^{-R},$$

where

$$
\begin{aligned}
&\Lambda = A_\infty, \quad R = R_1 + R_2 + \cdots ,\\
&(R_1)_{ij} = \begin{cases} (B_1)_{ij}, & \lambda_i = \lambda_j + 1,\\ 0, & \text{otherwise}, \end{cases}\\
&(\Psi_1)_{ij} = \begin{cases} -\dfrac{(B_1)_{ij}}{\lambda_i - \lambda_j - 1}, & \lambda_i \neq \lambda_j + 1,\\ \text{arbitrary}, & \text{otherwise}, \end{cases}\\
&B_1 = -\textstyle\sum_k A_k u_k,\\
&(R_2)_{ij} = \begin{cases} (B_2 - \Psi_1 R_1 + B_1 \Psi_1)_{ij}, & \lambda_i = \lambda_j + 2,\\ 0, & \text{otherwise}, \end{cases}\\
&(\Psi_2)_{ij} = \begin{cases} \dfrac{(-B_2 + \Psi_1 R_1 - B_1 \Psi_1)_{ij}}{\lambda_i - \lambda_j - 2}, & \lambda_i \neq \lambda_j + 2,\\ \text{arbitrary}, & \text{otherwise}, \end{cases}\\
&B_2 = -\sum_k A_k u_k^2,
\end{aligned}
\tag{2.11}
$$

etc.

It is not difficult to describe the ambiguity in the choice of the admissible pair of matrices Λ, R describing the local monodromy data of the system (2.4). Namely, the diagonal matrix Λ is defined up to permutations of its diagonal entries. Assuming the order fixed, the ambiguity in the choice of R can be described as follows [6]. Denote $\mathcal{C}_0(\Lambda) \subset \mathrm{GL}(V)$ the subgroup consisting of invertible linear operators $G\colon V \to V$ satisfying

$$z^\Lambda G\, z^{-\Lambda} = G_0 + z\, G_1 + z^2 G_2 + \cdots . \tag{2.12}$$

The definition of this subgroup can be reformulated [6] in terms of invariance of certain flag in V naturally associated with the semisimple operator Λ. The matrix $\tilde{R}$ obtained from R by the conjugation of the form

$$\tilde{R} = G^{-1} R\, G \tag{2.13}$$

will be called *equivalent* to R. Multiplying (2.10) on the right by G, one obtains another fundamental matrix solution to the same system of the same structure

$$\widetilde{\Phi}(z) := \Psi(z) z^\Lambda z^R G = \widetilde{\Psi}(z) z^\Lambda z^{\tilde{R}}$$

i.e., $\widetilde{\Psi}(z)$ is analytic at $z = 0$ with $\det \widetilde{\Psi}(0) \neq 0$.

The columns of the fundamental matrix (2.10) form a distinguished basis in the space of solutions to (2.4).

Definition 2.3. The basis given by the columns of the matrix (2.10) is called *Levelt basis* in the space of solutions to (2.4). The fundamental matrix (2.10) is called *Levelt fundamental matrix solution*.

The monodromy transformation of the Levelt fundamental matrix solution reads

$$\Phi\big(z\,e^{2\pi i}\big) = \Phi(z)M, \quad M = e^{2\pi i\Lambda}e^{2\pi i R}. \tag{2.14}$$

To conclude this section let us denote $\mathcal{C}(\Lambda, R)$ the subgroup of invertible transformations of the form

$$\mathcal{C}(\Lambda, R) = \{G \in \mathrm{GL}(V) \mid z^\Lambda G\, z^{-\Lambda} = \textstyle\sum_{k\in\mathbb{Z}} G_k z^k \text{ and } [G, R] = 0\}. \tag{2.15}$$

The subgroups $\mathcal{C}(\Lambda, R)$ and $\mathcal{C}(\Lambda, \tilde{R})$ associated with equivalent matrices R and $\tilde{R}$ are conjugated. It is easy to see that this subgroup coincides with the centralizer of the monodromy matrix (2.14)

$$G \in \mathcal{C}(\Lambda, R) \quad \text{iff} \quad Ge^{2\pi i\Lambda}e^{2\pi i R} = e^{2\pi i\Lambda}e^{2\pi i R}G, \quad \det G \neq 0. \tag{2.16}$$

Denote

$$\mathcal{C}_0(\Lambda, R) \subset \mathcal{C}(\Lambda, R) \tag{2.17}$$

the subgroup consisting of matrices G such that the expansion (2.15) contains only non-negative powers of z. Multiplying the Levelt fundamental matrix (2.10) by a matrix $G \in \mathcal{C}_0(\Lambda, R)$ one obtains another Levelt solution to (2.4)

$$\Psi(z)z^\Lambda z^R G = \widetilde{\Psi}(z)z^\Lambda z^R. \tag{2.18}$$

In the next section we will see that the quotient $\mathcal{C}(\Lambda, R)/\mathcal{C}_0(\Lambda, R)$ plays an important role in the theory of monodromy preserving deformations.

2.2 Monodromy data and isomonodromic deformations of a Fuchsian system

Denote $\lambda_j^{(k)}$, $j = 1, \ldots, m$, the eigenvalues of the matrix A_k, $k = 1, \ldots, n, \infty$ where the matrix A_∞ is defined as

$$A_\infty := -\sum_{k=1}^n A_k.$$

For the sake of technical simplicity let us assume that

$$\lambda_i^{(k)} \neq \lambda_j^{(k)} \quad \text{for } i \neq j,\ k = 1, \ldots, n, \infty. \tag{2.19}$$

Moreover, it will be assumed that A_∞ is a constant diagonal $m \times m$ matrix with eigenvalues $\lambda_j^{(\infty)}$, $j = 1, \ldots, m$.

Denote $\Lambda^{(k)}$, $R^{(k)}$ the local monodromy data of the Fuchsian system near the points $z = u_k$, $k = 1, \ldots, n, \infty$. The matrices $\Lambda^{(k)}$ are all diagonal

$$\Lambda^{(k)} = \mathrm{diag}(\lambda_1^{(k)}, \ldots, \lambda_m^{(k)}), \quad k = 1, \ldots, n, \infty. \tag{2.20}$$

and, under our assumptions

$$\Lambda^{(\infty)} = A_\infty.$$

Recall that the matrix $G \in \mathrm{GL}(m, \mathbb{C})$ belongs to the group $\mathcal{C}_0(\Lambda^{(\infty)})$ *iff*

$$z^{-\Lambda^{(\infty)}} G\, z^{\Lambda^{(\infty)}} = G_0 + \frac{G_1}{z} + \frac{G_2}{z^2} + \cdots . \tag{2.21}$$

It is easy to see that our assumptions about the eigenvalues of A_∞ imply diagonality of the matrix G_0.

Let us also remind that the matrices $\Lambda^{(k)}$ satisfy

$$\operatorname{Tr} \Lambda^{(1)} + \cdots + \operatorname{Tr} \Lambda^{(\infty)} = 0. \tag{2.22}$$

Definition 2.4. The numbers $\lambda_1^{(k)}, \ldots, \lambda_m^{(k)}$ are called the *exponents* of the system (1.2) at the singular point u_k.

Let us fix a fundamental matrix solutions of the form (2.10) near all singular points $u_1, \ldots, u_n, \infty$. To this end we are to fix branch cuts on the complex plane and choose the branches of logarithms $\log(z - u_1), \ldots, \log(z - u_n)$, $\log z^{-1}$. We will do it in the following way: perform parallel branch cuts π_k between ∞ and each of the u_k, $k = 1, \ldots, n$ along a given (generic) direction. After this we can fix Levelt fundamental matrices analytic on

$$z \in \mathbb{C} \setminus \bigcup_{k=1}^{n} \pi_k, \tag{2.23}$$

$$\Phi_k(z) = T_k\, (\mathbb{1} + \mathcal{O}(z - u_k))\, (z - u_k)^{\Lambda^{(k)}} (z - u_k)^{R^{(k)}}, \quad z \to u_k,\ k = 1, \ldots, n \tag{2.24}$$

and

$$\Phi(z) \equiv \Phi_\infty(z) = (\mathbb{1} + \mathcal{O}(1/z))\, z^{-A_\infty} z^{-R^{(\infty)}} \quad \text{as } z \to \infty, \tag{2.25}$$

Define the *connection matrices* by

$$\Phi_\infty(z) = \Phi_k(z) C_k, \tag{2.26}$$

where $\Phi_\infty(z)$ is to be analytically continued in a vicinity of the pole u_k along the positive side of the branch cut π_k.

The monodromy matrices M_k, $k = 1, \ldots, n, \infty$ are defined with respect to a basis $l_1, \ldots, l_n$ of loops in the fundamental group

$$\pi_1\left(\mathbb{C} \setminus \{u_1, \ldots u_n\}, \infty\right).$$

Choose the basis in the following way. The loop l_k arrives from infinity in a vicinity of u_k along one side of the branch cut π_k that will be called *positive*, then it encircles u_k going in anti-clock-wise direction leaving all other poles outside and, finally it returns to infinity along the opposite side of the branch cut π_k called *negative*.

Denote $l_j^* \Phi_\infty(z)$ the result of analytic continuation of the fundamental matrix $\Phi_\infty(z)$ along the loop l_j. The monodromy matrix M_j is defined by

$$l_j^* \Phi_\infty(z) = \Phi_\infty(z) M_j, \quad j = 1, \ldots, n. \tag{2.27}$$

The monodromy matrices satisfy

$$M_\infty M_n \dots M_1 = \mathbb{1}, \quad M_\infty = \exp\left(2\pi i A_\infty\right)\exp\left(2\pi i R^{(\infty)}\right) \tag{2.28}$$

if the branch cuts $\pi_1, \dots, \pi_n$ enter the infinite point according to the order of their labels, i.e., the positive side of π_{k+1} looks at the negative side of π_k, $k = 1, \dots, n-1$.

Clearly one has

$$M_k = C_k^{-1} \exp\left(2\pi i \Lambda^{(k)}\right)\exp\left(2\pi i R^{(k)}\right) C_k, \quad k = 1, \dots, n. \tag{2.29}$$

The collection of the local monodromy data $\Lambda^{(k)}$, $R^{(k)}$ together with the central connection matrices C_k will be used in order to uniquely fix the Fuchsian system with given poles. They will be defined up to an equivalence that we now describe. The eigenvalues of the diagonal matrices $\Lambda^{(k)}$ are defined up to permutations. Fixing the order of the eigenvalues, we define the class of equivalence of the nilpotent part $R^{(k)}$ and of the connection matrices C_k by factoring out the transformations of the form

$$\begin{gathered} R_k \mapsto G_k^{-1} R_k G_k, \quad C_k \mapsto G_k^{-1} C_k G_\infty, \quad k = 1, \dots, n, \\ G_k \in \mathcal{C}_0(\Lambda^{(k)}), \quad G_\infty \in \mathcal{C}_0(\Lambda^{(\infty)}). \end{gathered} \tag{2.30}$$

Observe that the monodromy matrices (2.29) will transform by a simultaneous conjugation

$$M_k \mapsto G_\infty^{-1} M_k G_\infty, \quad k = 1, 2, \dots, n, \infty.$$

Definition 2.5. The class of equivalence (2.30) of the collection

$$\Lambda^{(1)}, R^{(1)}, \dots, \Lambda^{(\infty)}, R^{(\infty)}, C_1, \dots, C_n \tag{2.31}$$

is called *monodromy data* of the Fuchsian system with respect to a fixed ordering of the eigenvalues of the matrices $A_1, \dots, A_n$ and a given choice of the branch cuts.

Lemma 2.6. *Two Fuchsian systems of the form* (1.2) *with the same poles* $u_1, \dots, u_n$, ∞ *and the same matrix* A_∞ *coincide, modulo diagonal conjugations if and only if they have the same monodromy data with respect to the same system of branch cuts* $\pi_1, \dots, \pi_n$.

Proof. Let

$$\Phi_\infty^{(1)}(z) = (\mathbb{1} + \mathcal{O}(1/z))\, z^{-\Lambda^{(\infty)}} z^{-R^{(\infty)}}, \quad \Phi_\infty^{(2)}(z) = (\mathbb{1} + \mathcal{O}(1/z))\, z^{-\tilde\Lambda^{(\infty)}} z^{-\tilde R^{(\infty)}}$$

be the fundamental matrices of the form (2.25) of the two Fuchsian systems. Using assumption about A_∞ we derive that $\tilde\Lambda^{(\infty)} = \Lambda^{(\infty)}$. Multiplying $\Phi_\infty^{(2)}(z)$ if necessary on the right by a matrix $G \in \mathcal{C}_0(\Lambda^{(\infty)})$, we can obtain another fundamental matrix of the second system with

$$\tilde R^{(\infty)} = R^{(\infty)}.$$

Consider the following matrix:

$$Y(z) := \Phi_\infty^{(2)}(z)[\Phi_\infty^{(1)}(z)]^{-1}. \tag{2.32}$$

$Y(z)$ is an analytic function around infinity:

$$Y(z) = G_0 + \mathcal{O}(1/z) \quad \text{as } z \to \infty \tag{2.33}$$

where G_0 is a diagonal matrix. Since the monodromy matrices coincide, $Y(z)$ is a single valued function on the punctured Riemann sphere $\overline{\mathbb{C}} \setminus \{u_1, \ldots, u_n\}$. Let us prove that $Y(z)$ is analytic also at the points u_k. Indeed, having fixed the monodromy data, we can choose the fundamental matrices $\Phi_k^{(1)}(z)$ and $\Phi_k^{(2)}(z)$ of the form (2.24) with the same connection matrices C_k and the same matrices $\Lambda^{(k)}$, $R^{(k)}$. Then near the point u_k, $Y(z)$ is analytic:

$$Y(z) = T_k^{(2)} (\mathbb{1} + \mathcal{O}(z - u_k)) \left[T_k^{(1)} (\mathbb{1} + \mathcal{O}(z - u_k))\right]^{-1}. \tag{2.34}$$

This proves that $Y(z)$ is an analytic function on all $\overline{\mathbb{C}}$ and then, by the Liouville theorem $Y(z) = G_0$, which is constant. So the two Fuchsian systems coincide, after conjugation by the diagonal matrix G_0. □

Remark 2.7. The connection matrices are determined, within their equivalence classes, by the monodromy matrices if the quotients

$$\mathcal{C}(\Lambda^{(k)}, R^{(k)})/\mathcal{C}_0(\Lambda^{(k)}, R^{(k)})$$

are trivial for all $k = 1, \ldots, n$. In particular this is the case when all the characteristic exponents at the poles $u_1, \ldots, u_n$ are non-resonant.

From the above lemma the following result readily follows.

Theorem 2.8. *If the matrices $A_k(u_1, \ldots, u_n)$ satisfy Schlesinger equations* (1.1) *and the matrix*

$$A_\infty = -(A_1 + \cdots + A_n)$$

is diagonal then all the characteristic exponents do not depend on $u_1, \ldots, u_n$. The fundamental matrix $\Phi_\infty(z; u)$ can be chosen in such a way that the nilpotent matrix $R^{(\infty)}$ and also all the monodromy matrices are constant in $u_1, \ldots, u_n$. The coefficients of expansion of the fundamental matrix in $1/z$ belong to a Picard–Vessiot type extension of the field $\mathcal{K}_{(n,m)}$ associated with the solution to Schlesinger equations. Moreover, the Levelt fundamental matrices $\Phi_k(z; u)$ can be chosen in such a way that all the nilpotent matrices $R^{(k)}$ and also all the connection matrices $\mathcal{C}_k$ are constant. Vice-versa, if the deformation $A_k = A_k(u_1, \ldots, u_n)$ is such that the monodromy data do not depend on $u_1, \ldots, u_n$ then the matrices $A_k(u_1, \ldots, u_n)$, $k = 1, \ldots, n$ satisfy Schlesinger equations.

Recall that the u-dependence of the needed fundamental matrix $\Phi_\infty(z; u)$ is to be determined from the linear equations

$$\partial_i \Phi_\infty(z; u) = -\frac{A_i}{z - u_i} \Phi_\infty(z; u), \quad i = 1, \ldots, n, \tag{2.35}$$

so

$$\partial_i \Psi_1 = -A_i, \tag{2.36}$$

$$\partial_i \Psi_2 = -A_i \Psi_1 - u_i A_i, \tag{2.37}$$

etc.

Example 2.9. The following example shows that in general the coefficients of expansion of the fundamental matrix may not be in the field $\mathcal{K}_{(n,m)}$. Indeed, let us consider the following isomonodromic deformation of the Fuchsian system

$$\frac{d\Phi}{dz} = \left[\frac{A_1}{z} + \frac{A_2}{z-x} + \frac{A_3}{z-1}\right]\Phi,$$

$$A_1 = \begin{pmatrix} -\frac{(\sqrt{x}+1)^2}{16\sqrt{x}} & -\frac{1}{2\sqrt{x}} \\ \frac{(\sqrt{x}+1)^4}{128\sqrt{x}} & \frac{(\sqrt{x}+1)^2}{16\sqrt{x}} \end{pmatrix},$$

$$A_2 = \begin{pmatrix} -\frac{3\sqrt{x}-1}{16\sqrt{x}} & \frac{1}{2(\sqrt{x}+1)\sqrt{x}} \\ -\frac{(\sqrt{x}+1)(3\sqrt{x}-1)^2}{128\sqrt{x}} & \frac{3\sqrt{x}-1}{16\sqrt{x}} \end{pmatrix},$$

$$A_3 = \begin{pmatrix} \frac{1}{16}\left(\sqrt{x}-3\right) & \frac{1}{2(\sqrt{x}+1)} \\ -\frac{1}{128}\left(\sqrt{x}-3\right)^2\left(\sqrt{x}+1\right) & \frac{1}{16}\left(3-\sqrt{x}\right) \end{pmatrix}.$$

In this case

$$A_\infty = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad R^{(\infty)} = R_1^{(\infty)} = \begin{pmatrix} 0 & -\frac{1}{2} \\ 0 & 0 \end{pmatrix}.$$

The fundamental matrix

$$\Phi = \left(\mathbb{1} + \frac{\Psi_1}{z} + \mathcal{O}\left(1/z^2\right)\right) z^{-A_\infty} z^{-R}$$

satisfying also the equation

$$\frac{\partial \Phi}{\partial x} = -\frac{A_2}{z-x}\Phi$$

has

$$\Psi_1 = \begin{pmatrix} \frac{1}{16}\left(3x - 2\sqrt{x}\right) & -\log\left(\sqrt{x}+1\right) \\ \frac{1}{128}\left(\frac{9x^2}{2} + 2x^{3/2} - 5x + 2\sqrt{x}\right) & \frac{1}{16}\left(2\sqrt{x} - 3x\right) \end{pmatrix}.$$

This matrix does not belong to the field $\mathcal{K}_{3,2}$ isomorphic in this case to the field of rational functions in $\sqrt{x}$.

3 Reductions of the Schlesinger systems

3.1 Reducible monodromy groups

Definition 3.1. Given a Fuchsian system of the form (1.2), we say that its monodromy group $\langle M_1, \ldots, M_n \rangle$ is *l-reducible,* $0 < l < m$ if the monodromy matrices admit a common invariant subspace X_l of dimension l in the space of solutions of the system (1.2).

In particular, if the monodromy group is l-reducible, then there exists a basis where all monodromy matrices have the form

$$M_k = \left(\begin{array}{c|c} \delta_k & \beta_k \\ \hline 0 & \gamma_k \end{array}\right), \quad k = 1, \ldots, n, \infty,$$

where δ_k, β_k and γ_k are respectively some $l \times l$, $l \times (m-l)$ and $(m-l) \times (m-l)$ matrices.

Given the above definition, we can proceed to the proof of Theorem 1.1.

We begin with the proof of Lemma 1.3. Our proof, valid for the case of diagonalizable A_∞, is based on the fact that the sum of the exponents of the invariant subspace X_l must always be a negative integer (see [2], Lemma 5.2.2). We will perform a sequence of gauge transformations which map such sum to zero. Let $\lambda_1^{(\infty)}, \ldots, \lambda_m^{(\infty)}$ be the eigenvalues of A_∞ (which is assumed to be diagonal). By means of a permutation $P \in S_m$, we order the eigenvalues of A_∞ as follows: the first l eigenvalues correspond to the invariant subspace X_l and we order them in such a way that $\operatorname{Re}\lambda_1 \geq \operatorname{Re}\lambda_r$, for all $r = 2, \ldots, l$. Then we order the other eigenvalues in such a way that $\operatorname{Re}\lambda_m \leq \operatorname{Re}\lambda_s$ for all $s = l+1, \ldots, m-1$.

Let us fix a fundamental matrix Φ normalized at infinity

$$\Phi_\infty = \left(\mathbb{1} + \frac{\Psi_1}{z} + \frac{\Psi_2}{z^2} + \mathcal{O}\big(1/z^3\big)\right) z^{-A_\infty} z^{-R^{(\infty)}},$$

where Ψ_1, Ψ_2 and $R^{(\infty)}$ are given by formulae (2.11) in Example 2.2.

Consider the following gauge transformation $\Phi(z) = (I(z) + G)\widetilde{\Phi}(z)$ where

$$\begin{aligned}
&I(z) := \operatorname{diag}(z, 0, \ldots, 0), \quad \text{and} \\
&G_{m1} = \Psi_{1_{m1}}, \quad G_{1m} = -\frac{1}{G_{m1}}, \\
&\text{if } p \neq 1, m, \quad G_{pp} = 1, \quad G_{1p} = \Psi_{1_{mp}} G_{1m}, \quad G_{p1} = \Psi_{1_{p1}}, \\
&\text{if } p, q \neq 1,\ p \neq q, \quad G_{pq} = 0, \\
&G_{11} = G_{1m}\Psi_{2_{m1}} + \Psi_{1_{11}}, \quad \text{and} \quad G_{mm} = 0.
\end{aligned} \tag{3.1}$$

Let us first observe that the entries of the matrix G belong to an extension of the differential field $\mathcal{K}_{(n,m)}$ obtained by adding solutions of the linear equations (2.36), (2.37). In order to see that this gauge transformation always works let us show that $\Psi_{1_{m1}}(u)$ is never identically equal to zero if at least one of the $(m, 1)$ matrix entries of the matrices $A_1(u), \dots, A_n(u)$ is not identically zero. Indeed, this follows from the equations (2.36).

Let us prove that this transformation maps the matrices $A_1, \dots, A_n$ to new matrices $\tilde{A}_1, \dots, \tilde{A}_n$ given by

$$\tilde{A}_k := (I(u_k) + G)^{-1} A_k (I(u_k) + G),$$

such that

$$\tilde{A}_\infty = -\sum_{k=1}^{n} \tilde{A}_k = \operatorname{diag}\left(\lambda_1^{(\infty)} + 1, \lambda_2^{(\infty)}, \dots, \lambda_{m-1}^{(\infty)}, \lambda_m^{(\infty)} - 1\right). \tag{3.2}$$

In fact $(I(z) + G)^{-1} = J(z) + G^{-1}$ where

$$J(z) := \operatorname{diag}(0, \dots, 0, z),$$

therefore

$$\tilde{A}_k := G^{-1} A_k I(u_k) + G^{-1} A_k G + J(u_k) A_k I(u_k) + J(u_k) A_k G.$$

Multiplying by G from the left and summing on all k we get that the condition (3.2) is satisfied if and only if

$$\begin{pmatrix} -g_{11} & (\lambda_1^{(\infty)} - \lambda_2^{(\infty)}) g_{12} & \dots & (\lambda_1^{(\infty)} - \lambda_{m-1}^{(\infty)}) g_{1\,m-1} & (\lambda_1^{(\infty)} - \lambda_m^{(\infty)} + 1) g_{1m} \\ (\lambda_2^{(\infty)} - \lambda_1^{(\infty)} - 1) g_{21} & 0 & \dots & \dots & 0 \\ \dots & 0 & \dots & \dots & 0 \\ (\lambda_m^{(\infty)} - \lambda_1^{(\infty)} - 1) g_{m1} & 0 & \dots & \dots & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \sum_k A_{k_{11}} u_k & 0 \dots 0 \\ \dots & 0 \dots 0 \\ \sum_k A_{k_{m1}} u_k & 0 \dots 0 \end{pmatrix} + \begin{pmatrix} g_{1m} \sum_k A_{k_{m1}} u_k^2 & 0 \dots 0 \\ 0 & 0 \dots 0 \\ \dots & 0 \dots 0 \end{pmatrix}$$

$$+ \begin{pmatrix} g_{1m} \sum_s \sum_k A_{k_{ms}} u_k g_{s1} & \dots & g_{1m} \sum_s \sum_k A_{k_{ms}} u_k g_{sm} \\ 0 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & 0 \end{pmatrix}. \tag{3.3}$$

Observe that in the non-resonant case, these formulae are clearly satisfied thanks to the fact that Ψ_1, Ψ_2 and $R^{(\infty)}$ are given by formulae (2.11) in Example 2.2. In the resonant case, we only need to prove that when there is a resonance of type $\lambda_m^{(\infty)} - \lambda_p^{(\infty)} = 1$ or $\lambda_p^{(\infty)} - \lambda_1^{(\infty)} = 1$ for any $p = 1, \dots, m-1$, then the corresponding coefficients $\sum_k A_{k_{mp}} u_k$, and $\sum_k A_{k_{p1}} u_k$ are zero. Observe that such entries coincide with the (m, p) and $(p, 1)$ entries in the matrix $R_1^{(\infty)}$ defined in Section 2.1 (see the formulae (2.11)). Due to our ordering of the eigenvalues, if $\lambda_m^{(\infty)} - \lambda_p^{(\infty)} = 1$ then $p = 1, \dots, l$ and if $\lambda_p^{(\infty)} - \lambda_1^{(\infty)} = 1$ then $p = l+1, \dots, m$. This means that the corresponding $R_1^{(\infty)}$ must lie in the $l \times (m-l)$ lower left block, which is 0 by the hypothesis that the monodromy group is l-reducible.

Finally, if $\lambda_m^{(\infty)} - \lambda_1^{(\infty)} = 2$, we find that the gauge transformation works only if

$$\Big(\sum_{l=1}^{n} A_{l_{m1}} u_l\Big)\Big(\sum_{l=1}^{n}(A_{l_{11}} - A_{l_{mm}})u_l\Big) - \sum_{l=1}^{n} A_{l_{m1}} u_l^2 - \sum_{p=2}^{m-1}\Big(\sum_{l=1}^{n} A_{l_{mp}} u_l\Big) G_{p1} = 0.$$

This is precisely the condition $\big(R_2^{(\infty)}\big)_{m1} = 0$, as it follows from (2.11).

Let us prove that this gauge transformation preserves the Schlesinger equations. Differentiating $\tilde{A}_k$ w.r.t. u_j, with $j \neq k$ and using the Schlesinger equations for $A_1, \dots, A_n$ we get

$$\begin{aligned}\frac{\partial \tilde{A}_k}{\partial u_j} &= \left[\tilde{A}_k, (I(u_k)+G)^{-1}\frac{\partial G}{\partial u_j} + \frac{(I(u_k)+G)^{-1}A_j(I(u_k)+G)}{u_k-u_j}\right]\\ &= \frac{\big[\tilde{A}_k, \tilde{A}_j\big]}{u_k-u_j}\\ &\quad + \left[\tilde{A}_k, (I(u_k)+G)^{-1}\left(\frac{\partial G}{\partial u_j} + \frac{A_j(I(u_k)-I(u_j)) - B_{kj}A_j(I(u_j)+G)}{u_k-u_j}\right)\right],\end{aligned}$$

where

$$B_{kj} = \begin{pmatrix} 0 & \dots & 0 & \frac{u_k-u_j}{g_{m1}}\\ 0 & \dots & 0 & 0\\ \dots & \dots & \dots & \dots\\ 0 & \dots & 0 & 0\end{pmatrix}.$$

Given the formulae (3.1), it is straightforward to prove that the equation

$$\frac{\partial G}{\partial u_j} + \frac{A_j(I(u_k)-I(u_j)) - B_{kj}A_j(I(u_j)+G)}{u_k-u_j} = 0,$$

is equivalent to the equations (2.36), (2.37). This proves that also $\tilde{A}_1, \dots, \tilde{A}_n$ satisfy the Schlesinger equations.

Now let the sum of the exponents of the invariant subspace X_l be $-N$, where N is a positive integer. By iterating the above gauge transformation N times, we arrive at a new solution $(B_1, \dots, B_n)$ of the Schlesinger equations $S_{(n,m)}$ such that the sum of the exponents of the invariant subspace X_l is zero and

$$B_\infty = \operatorname{diag}\big(\lambda_1^{(\infty)} + N, \lambda_2^{(\infty)}, \dots, \lambda_{m-1}^{(\infty)}, \lambda_m^{(\infty)} - N\big).$$

To conclude the proof of this lemma, let us prove that this new solution $(B_1, \dots, B_n)$ is of the form

$$B_{k_{ij}} = 0 \quad \text{for all } i = l+1, \dots, n,\ j = 1, \dots, l.$$

In fact suppose by contradiction that B_k are not in the above form. Then by Lemma 5.2.2. in [2], there exists a gauge transformation P, constant in z, such that the new residue matrices $\tilde{B}_k = P^{-1}B_kP$ have the form has the form

$$\tilde{B}_{k_{ij}} = 0 \quad \text{for all } i = l+1, \dots, n,\ j = 1, \dots, l.$$

In general $\tilde{B}_\infty$ won't be diagonal, but we can diagonalize it by a constant gauge transformation Q preserving the block triangular form of $\tilde{B}_1, \dots, \tilde{B}_n$. So we end up with

$$\hat{B}_\infty = Q^{-1}P^{-1}B_\infty PQ, \quad \hat{B}_k = Q^{-1}P^{-1}B_k PQ,$$

and since $Q^{-1}P^{-1}B_\infty PQ = B_\infty$, we have that PQ is diagonal. But then if B_k is not block upper triangular, $\hat{B}_k$ is not either, so we obtain a contradiction. Lemma 1.3 is proved. $\square$

Proof of Theorem 1.1. By Lemma 1.3, we obtained a gauge transformation mapping a solution $(A_1, \dots, A_n)$ of the Schlesinger system $S_{n,m}$ with an l-reducible monodromy group to a solution $B_1, \dots, B_n$ of the block triangular form. As it was explained in the Introduction, the solution $(B_1(u), \dots, B_n(u))$ belongs to a Picard–Vessiot type extension $\mathcal{K}^{(N)}$ for some N of the composite

$$\mathcal{K} = \mathcal{K}_{n,l}\mathcal{K}_{n,m-l}.$$

So, to conclude the proof of this theorem, we need to prove Lemma 1.4.

Let us prove the formulae (1.6). Our gauge transformation constructed in Lemma 1.3 is an iteration of elementary gauges transformation $\Phi = (I(z) + G)\widetilde{\Phi}$ mapping the matrices $A_1, \dots, A_n$ to new matrices $\tilde{A}_1, \dots, \tilde{A}_n$ such that $\tilde{A}_\infty = A_\infty + \mathrm{diag}(1, 0, \dots, 0, -1)$.

Let us prove that each elementary gauge transformation preserves the normalization at infinity. More precisely, we prove that if we fix a fundamental matrix Φ normalized at infinity

$$\Phi_\infty = \left(\mathbb{1} + \frac{\Psi_1}{z} + \frac{\Psi_2}{z^2} + \mathcal{O}\big(1/z^3\big)\right) z^{-A_\infty} z^{-R^{(\infty)}},$$

then $\widetilde{\Phi} = (J(z) + G^{-1})\Phi_\infty = (\mathbb{1} + \mathcal{O}(1/z))\, z^{-\tilde{A}_\infty} z^{-\tilde{R}^{(\infty)}}$, with $\tilde{R}^{(\infty)} = R^{(\infty)}$.

In fact it is straightforward to prove that

$$(J(z) + G^{-1})\left(\mathbb{1} + \frac{\Psi_1}{z} + \frac{\Psi_2}{z^2} + \mathcal{O}\big(1/z^3\big)\right) \mathrm{diag}(z, 0, \dots, 0, 1/z)$$
$$= \chi_1 z + \chi_0 + \mathcal{O}\,(1/z)$$

where all matrix elements of χ_1 are zero apart from the $(m, 1)$ element which is

$$\chi_{1_{m1}} = \frac{1}{g_{1m}} + \Psi_{1_{m1}}$$

and the matrix elements of χ_0 are given by the following: for $p \neq 1, m$

$$\chi_{0_{pp}} = 1, \quad \chi_{0_{p1}} = -g_{p1} + \Psi_{1_{p1}}, \quad \chi_{0_{pm}} = -\frac{g_{1p}}{g_{1m}} + \Psi_{1_{mp}},$$

$$\chi_{0_{11}} = \chi_{0_{mm}} = 1,$$

and

$$\chi_{0_{m1}} = -\frac{g_{11} - \sum_{p=2}^{m-1} g_{1p}\Psi_{1_{pm}} - \Psi_{1_{11}} + \sum_{p=2}^{m-1} g_{1p}\Psi_{1_{p1}}}{g_{1m}} + \Psi_{2_{1m}}.$$

Using the formulae (3.1) for G it is easy to prove that all entries of χ_0 and χ_1 are zero.

Therefore each elementary gauge transformation preserves the normalization at infinity and maps A_∞ to

$$\tilde{A}_\infty = A_\infty + \mathrm{diag}(1, 0, \ldots, 0, -1).$$

Since the fundamental matrix remains normalized at infinity and the gauge transformation $\Phi = (I(z) + G)\widetilde{\Phi}$ is analytic over $\mathbb{C}$, all monodromy data $\Lambda^{(1)}(A)$, $R^{(1)}(A), \ldots, \Lambda^{(n)}(A)$, $R^{(n)}(A)$, $C_1(A), \ldots, C_n(A)$ are preserved in each iteration. Finally we prove that $R^{(\infty)} = \tilde{R}^{(\infty)}$. Due to the above we only need to prove that if

$$z^{-A_\infty} R^{(\infty)}(A) z^{A_\infty} = \frac{R_1}{z} + \frac{R_2}{z^2} + \cdots$$

where $R_1, R_2, \ldots$ are some matrices defined in Section 2, then the matrix

$$z^{-B_\infty} R^{(\infty)}(A) z^{B_\infty}$$

is also polynomial in $1/z$. Since $B_\infty = A_\infty + \mathrm{diag}(N, 0, \ldots, 0, -N)$ we get

$$\begin{aligned} &z^{-B_\infty} R^{(\infty)}(A) z^{B_\infty} \\ &\quad = \mathrm{diag}(z^{-N}, 1, \ldots, 1, z^N) z^{-A_\infty} R^{(\infty)}(A) z^{A_\infty} \,\mathrm{diag}(z^N, 1, \ldots, 1, z^{-N}) \\ &\quad = \mathrm{diag}(z^{-N}, 1, \ldots, 1, z^N)\left(\frac{R_1}{z} + \frac{R_2}{z^2} + \cdots\right)\mathrm{diag}(z^N, 1, \ldots, 1, z^{-N}) \\ &\quad = \mathrm{Pol}\,(1/z) + \mathrm{Div}(z), \end{aligned}$$

where $\mathrm{Pol}\,(1/z)$ and $\mathrm{Div}(z)$ are matrix values polynomials in $1/z$ and z respectively. The matrix elements of the latter are of the form:

$$\begin{gathered} \mathrm{Div}_{pq} = 0, \quad \text{for } q \neq 1, p \neq m, \quad \mathrm{Div}_{11} = 0, \quad \mathrm{Div}_{mm} = 0, \\ \mathrm{Div}_{p1} = \sum R^{(\infty)}_{k_{p1}} z^{N-k}, \quad \text{for } p \neq 1, m, \\ \mathrm{Div}_{mq} = \sum R^{(\infty)}_{k_{mq}} z^{N-k}, \quad \text{for } q \neq 1, m \\ \mathrm{Div}_{m1} = \sum R^{(\infty)}_{k_{m1}} z^{2N-k}. \end{gathered}$$

Since the monodromy group is reducible, all the entries of $R_k^{(\infty)}$ involved in the above expressions are identically zero. Therefore $\mathrm{Div}(z) \equiv 0$ as we wanted to prove. This proves the relations (1.6).

Let us now prove the statement ii) of Lemma 1.4. Starting form the solution $(B_1, \ldots, B_n)$, we can reconstruct $(A_1, \ldots, A_n)$ by iterating another gauge transformation of the form $(J(z)+F)$ where $J(z)=\mathrm{diag}(0,\dots,0,z)$ and

$$\begin{aligned}
&F_{1m}=\widetilde{\Psi}_{1_{1m}}, \quad F_{m1}=-\frac{1}{\widetilde{\Psi}_{1_{1m}}},\\
&\text{if } p\neq 1,m, \quad F_{pp}=1, \quad F_{mp}=\widetilde{\Psi}_{1_{1p}}G_{m1}, \quad F_{pm}=\widetilde{\Psi}_{1_{pm}},\\
&\text{if } p,q\neq 1,\ p\neq q, \quad F_{pq}=0,\\
&F_{11}=0, \quad \text{and} \quad F_{mm}=F_{m1}\Psi_{2_{1m}}+\Psi_{1_{mm}}.
\end{aligned} \tag{3.4}$$

This gauge transformation is always well defined because $\widetilde{\Psi}_{1_{1m}}$ is always non-zero (proof of this fact is analogous to the proof that $\Psi_{1_{m1}}$ is never zero given above). Following the same computations as in the proof of Lemmata 1.3 and 1.4, it is easy to verify that this gauge transformation preserves the Schlesinger equations, the normalization of the fundamental matrix at infinity, $R^{(\infty)}$ and maps $\tilde{A}_\infty$ to

$$A_\infty=\tilde{A}_\infty-\mathrm{diag}(1,0,\dots,0,-1).$$

The above arguments complete the proof of Lemma 1.4 and, therefore of Theorem 1.1. □

3.1.1 Upper triangular monodromy groups. In this section we deal with the case of upper triangular monodromy groups, that is there exists a basis where all monodromy matrices have the form

$$M_{k_{ij}}=0 \quad \text{for all } i>j.$$

To prove Corollary 1.2 we iterate the procedure of the proof of Lemma 1.3: at the first step we show that $(A_1,\dots,A_n)$ is mapped by a rational gauge transformation to $(A_1^{(1)},\dots,A_n^{(1)})$ of the form

$$A^{(1)}_{k\ i1}=0 \quad \text{for all } i\neq 1, k=1,\dots,n.$$

At the l-th step we show that $(A_1,\dots,A_n)$ is mapped by a rational gauge transformation to $(A_1^{(l)},\dots,A_n^{(l)})$ of the form

$$A^{(l)}_{k\ ij}=0, \quad i>j,\ j\leq l,\ k=1,\dots,n.$$

At the m-th step we obtain that is mapped by a rational gauge transformation to

$$\tilde{A}_{k_{ij}}:=A^{(m)}_{k_{ij}}=0 \quad \text{for all } i>j.$$

Let us show that $\tilde{A}_{k_{ij}}(u_1,\dots,u_n)$ belongs to the Picard–Vessiot type extension $\mathcal{K}^{(N)}$ for some N of

$$\mathcal{K}=\mathbb{C}(u_1,\dots,u_n).$$

Clearly the diagonal elements $\tilde{A}_k$ are the eigenvalues $\lambda_1^{(k)}, \dots \lambda_m^{(k)}$. The Schlesinger equations for $i \neq j$ read:

$$\frac{\partial}{\partial u_j}\tilde{A}_{i_{p,p+q}} = \frac{\lambda_{p+q}^{(j)} - \lambda_p^{(j)}}{u_i - u_j}\tilde{A}_{i_{p,p+q}} - \frac{\lambda_{p+q}^{(i)} - \lambda_p^{(i)}}{u_i - u_j}\tilde{A}_{j_{p,p+q}} + \sum_{s=1}^{q-1} \frac{\tilde{A}_{i_{p,p+s}}\tilde{A}_{j_{p+s,p+q}} - \tilde{A}_{j_{p,p+s}}\tilde{A}_{i_{p+s,p+q}}}{u_i - u_j}, \tag{3.5}$$

and for $i = j$

$$\frac{\partial}{\partial u_i}\tilde{A}_{i_{p,p+q}} = -\sum_{j\neq i}\left[\frac{\lambda_{p+q}^{(j)} - \lambda_p^{(j)}}{u_i - u_j}\tilde{A}_{i_{p,p+q}} - \frac{\lambda_{p+q}^{(i)} - \lambda_p^{(i)}}{u_i - u_j}\tilde{A}_{j_{p,p+q}} + \sum_{s=1}^{q-1} \frac{\tilde{A}_{i_{p,p+s}}\tilde{A}_{j_{p+s,p+q}} - \tilde{A}_{j_{p,p+s}}\tilde{A}_{i_{p+s,p+q}}}{u_i - u_j}\right],$$

where for $q = 1$ the sum $\sum_{s=1}^{q-1}$ is zero. It is clear that for each q, $1 \leq q < m - p$, the differential system for the matrix elements $A_{i_{p,p+q}}$, $i = 1, \dots, n$ is linear and it is Pfaffian integrable because the Schlesinger equations are Pfaffian integrable. In particular it is worth observing that for each q, $1 \leq q < m - p$, the homogeneous part of such differential system is the Lauricella hypergeometric system (see [16]).

3.2 An example

Consider the following solution A_1, A_2, A_3 of the Schlesinger equations in dimension $m = 2$, where we have chosen $u_1 = 0$, $u_2 = x$, $u_3 = 1$, with matrix entries:

$$A_{1_{11}} = \frac{2\log\frac{\sqrt{x}+1}{\sqrt{x}-1}\sqrt{x}(x^2+4x-5) - 4x(4+3x) - \left(\log\frac{\sqrt{x}+1}{\sqrt{x}-1}\right)^2 (x-1)^2}{2\left((x-1)\log\frac{\sqrt{x}+1}{\sqrt{x}-1} - 2\sqrt{x}\right)^2},$$

$$A_{1_{12}} = \frac{\left(\log\frac{\sqrt{x}+1}{\sqrt{x}-1}\right)^2 (x-1)^2 + 2x(5+3x) - (x^2+6x-7)\sqrt{x}\log\frac{\sqrt{x}+1}{\sqrt{x}-1}}{4\left((x-1)\log\frac{\sqrt{x}+1}{\sqrt{x}-1} - 2\sqrt{x}\right)^2 (x-1)} \times \left(6\sqrt{x}(1+x) - (x^2+2x-3)\log\frac{\sqrt{x}+1}{\sqrt{x}-1}\right),$$

$$A_{1_{21}} = \frac{4\sqrt{x}(1-x)}{\left((x-1)\log\frac{\sqrt{x}+1}{\sqrt{x}-1} - 2\sqrt{x}\right)^2},$$

$$A_{2_{11}} = \frac{\left(\log \frac{\sqrt{x}+1}{\sqrt{x}-1}\right)^2 (x-1)^3 - 4x(7+x) - 8\log \frac{\sqrt{x}+1}{\sqrt{x}-1}\sqrt{x}(x^2-3x+2)}{4\left((x-1)\log \frac{\sqrt{x}+1}{\sqrt{x}-1} - 2\sqrt{x}\right)^2 (x-1)},$$

$$A_{2_{21}} = -\frac{4\sqrt{x}}{\left((x-1)\log \frac{\sqrt{x}+1}{\sqrt{x}-1} - 2\sqrt{x}\right)^2},$$

$$A_{2_{12}} = -\frac{\left(\log \frac{\sqrt{x}+1}{\sqrt{x}-1}\right)^2 (x-1)^3 - 16x - 2(3x^2-8x+5)\sqrt{x}\log \frac{\sqrt{x}+1}{\sqrt{x}-1}}{8\left((x-1)\log \frac{\sqrt{x}+1}{\sqrt{x}-1} - 2\sqrt{x}\right)^2 (x-1)^2} \times \left(2\sqrt{x}(3+x) + (x^2-4x+3)\log \frac{\sqrt{x}+1}{\sqrt{x}-1}\right),$$

$$A_{3_{11}} = \frac{1}{2} - A_{1_{11}} - A_{2_{11}}, \quad A_{3_{12}} = -A_{1_{12}} - A_{2_{12}}, \quad A_{3_{21}} = -A_{1_{21}} - A_{2_{21}},$$
$$A_{1_{22}} = -A_{1_{11}}, \quad A_{2_{22}} = -A_{2_{11}}, \quad A_{3_{22}} = -A_{3_{11}}.$$

This solution has a reducible monodromy group. Observe that

$$A_\infty = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

is resonant. By applying our technique, it is straightforward to obtain a the new solution B_1, B_2, B_3 of the Schlesinger equations, gauge equivalent to A_1, A_2, A_3 in the upper triangular form:

$$B_1 = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{x}}{x-1} \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad B_2 = \begin{pmatrix} -\frac{1}{4} & \frac{\sqrt{x}}{(x-1)^2} \\ 0 & \frac{1}{4} \end{pmatrix}, \quad B_3 = \begin{pmatrix} -\frac{3}{4} & -\frac{x\sqrt{x}}{(x-1)^2} \\ 0 & \frac{3}{4} \end{pmatrix}, \quad B_\infty = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}.$$

This new solution is actually algebraic. This shows that the differential fields $\mathcal{K}^A_{3,2}$ and $\mathcal{K}^B_{3,2}$ associated with the solutions A_1, A_2, A_3 and B_1, B_2, B_3 respectively are not isomorphic.

3.3 Smaller monodromy groups

The proof of Theorem 1.5 is based on a few lemmata.

Lemma 3.2. *Let $A_1, \dots, A_n$ be a solution of the Schlesinger equations such that one of the monodromy matrices $(M_1, \dots, M_n)$, say M_l, is proportional to the identity, then there exists a solution $\tilde{A}_1, \dots, \tilde{A}_{l-1}, \tilde{A}_{l+1}, \dots, \tilde{A}_n$ of the Schlesinger equations in $n-1$ variables with monodromy matrices $M_1, \dots, M_{l-1}, M_{l+1}, \dots, M_n$. The original solution $A_1, \dots, A_n$ depends rationally on $\tilde{A}_1, \dots, \tilde{A}_{l-1}, \tilde{A}_{l+1}, \dots, \tilde{A}_n$, $\tilde{\Phi}(u_l)$ and on u_l.*

Proof. Let us first consider the case $M_l = \mathbb{1}$, for simplicity, $l = n$. This means that all eigenvalues $\lambda_1^{(n)}, \ldots \lambda_m^{(n)}$ of A_n are integers and $R^{(n)} = 0$. To eliminate the singularity n, we perform a conformal transformation $\zeta = \frac{1}{z-u_n}$. We obtain

$$\frac{\mathrm{d}\Phi}{\mathrm{d}\zeta} = \left(\frac{A_\infty}{\zeta} + \sum_{k=1}^{n-1} \frac{A_k}{\zeta - \tilde{u}_k}\right)\Phi,$$

where $\tilde{u}_k = \frac{1}{u_k - u_n}$, for $k \neq n$. The new residue matrix at infinity is $-A_n$. We perform a gauge transformation diagonalizing A_n and use iterations of the gauge transformation of the form $(I(\zeta) + G)$ where G is defined by formulae (3.3) to map all eigenvalues of A_n to zero.

We have seen in the proof of Lemma 1.3 that this gauge transformation is always well defined and it works for $R^{(\infty)} = 0$. Of course similar formulae can be given to map any $\lambda_j^{(\infty)}$ to $\lambda_j^{(\infty)} + 1$ and any $\lambda_i^{(\infty)}$ to $\lambda_i^{(\infty)} - 1$. After enough iterations we end up with a new Fuchsian system of the form

$$\frac{\mathrm{d}\widetilde{\Phi}}{\mathrm{d}\zeta} = \left(\frac{\tilde{A}_\infty}{\zeta} + \sum_{k=1}^{n-1} \frac{\tilde{A}_k}{\zeta - \tilde{u}_k}\right)\widetilde{\Phi},$$

such that the residue at infinity is $\tilde{A}_n = 0$.

Now we perform the inverse conformal transformation, $z = \frac{1}{\zeta} + u_n$, we obtain

$$\frac{\mathrm{d}\widetilde{\Phi}}{\mathrm{d}z} = \sum_{k=1}^{n-1} \frac{\tilde{A}_k}{z - u_k}\widetilde{\Phi},$$

and the residue at infinity is $\tilde{A}_\infty$. We finally perform a gauge transformation diagonalizing $\tilde{A}_\infty$, so that the final Fuchsian system is

$$\frac{\mathrm{d}\widehat{\Phi}}{\mathrm{d}z} = \sum_{k=1}^{n-1} \frac{\hat{A}_k}{z - u_k}\widehat{\Phi},$$

where $\hat{A}_\infty = A_\infty$.

All the monodromy data of this new system coincide with the ones of the original system with matrices $A_1, \ldots, A_n, A_\infty$. The proof of this fact is very similar to the proof of statement (ii) of Lemma 1.4 and we omit it.

The new matrices $\hat{A}_1, \ldots, \hat{A}_{n-1}$ satisfy the Schlesinger equations because the gauge transformations of the form $(I(z) + G)$ where G is defined by the formulae (3.3) preserve the Schlesinger equations. Observe that since $\hat{A}_n$ is zero, $\hat{A}_1, \ldots, \hat{A}_{n-1}$ satisfy the Schlesinger equations $S_{n-1,m}$.

We now want to reconstruct the original solution $A_1, \ldots, A_n$ from $\hat{A}_1, \ldots, \hat{A}_{n-1}$.

Let us consider the Fuchsian system

$$\frac{d\widehat{\Phi}}{dz} = \sum_{k=1}^{n-1} \frac{\hat{A}_k}{z - u_k} \widehat{\Phi}.$$

Let us choose any point $u_n \neq u_k$, $k = 1, \dots, n-1$ and perform the constant gauge transformation $\widehat{\Psi} = \widehat{\Phi}(u_n)^{-1}\widehat{\Phi}$, where $\widehat{\Phi}(u_n)$ is the value at $z = u_n$ of

$$\widehat{\Phi}(z) = (\mathbb{1} + \mathcal{O}(1/z))\, z^{-A_\infty} z^{R^{(\infty)}}.$$

Let us perform the conformal transformation $\zeta = \frac{1}{z-u_n}$,

$$\widehat{\Psi}(\zeta) := \widehat{\Phi}(u_n)^{-1}\widehat{\Phi}\Big(\frac{1}{\zeta} + u_n\Big).$$

Let us apply a product $F_\infty(\zeta)$ of gauge transformations of the form $(J(\zeta) + F)$, where F is given by the formulae (3.4), to create a new non-zero residue matrix at infinity with integer entries $\lambda_1^{(n)}, \dots \lambda_m^{(n)}$:

$$\widehat{\Psi}(\zeta) = F_\infty(\zeta)\widehat{\Psi}(\zeta) = F_\infty(\zeta)\widehat{\Phi}(u_n)^{-1}\widehat{\Phi}\Big(\frac{1}{\zeta} + u_n\Big).$$

Let us now apply the conformal transformation $z = \frac{1}{\zeta} + u_n$:

$$\widetilde{\Phi}(z) := F_\infty\Big(\frac{1}{z - u_n}\Big)\widehat{\Phi}(u_n)^{-1}\widehat{\Phi}(z).$$

We need now to diagonalize the new residue matrix at infinity

$$\tilde{A}_\infty = F_\infty(u_n)\widehat{\Phi}(u_n)^{-1}\hat{A}_\infty\widehat{\Phi}(u_n)F_\infty(u_n)^{-1}.$$

To do so we put

$$\Phi(z) := \widehat{\Phi}(u_n)F_\infty(u_n)^{-1}\widehat{\Phi}(z).$$

The new residue matrices are

$$\begin{aligned} B_i = {} & \widehat{\Phi}(u_n)F_\infty(u_n)^{-1}F_\infty\Big(\frac{1}{u_i - u_n}\Big)\widehat{\Phi}(u_n)^{-1}\hat{A}_i \\ & \cdot \widehat{\Phi}(u_n)F_\infty\Big(\frac{1}{u_i - u_n}\Big)^{-1}F_\infty(u_n)\widehat{\Phi}(u_n)^{-1} \end{aligned}$$

for $i = 1, \dots, n-1$ and

$$B_n = \widehat{\Phi}(u_n)F_\infty^{-1}(u_n) \cdot \operatorname{diag}(\lambda_1^{(n)}, \dots, \lambda_m^{(n)})F_\infty(u_n)\widehat{\Phi}(u_n)^{-1}.$$

The Fuchsian system with residue matrices $B_1, \dots, B_n, B_\infty$ has the same exponents and the same monodromy data as the original system of residue matrices $A_1, \dots, A_n$, A_∞. Therefore, by the uniqueness Lemma 2.6, $A_1, \dots, A_n, A_\infty$ coincide with $B_1, \dots, B_n, B_\infty$ up to diagonal conjugation.

As a consequence $A_1, \dots, A_n$ depend rationally on $\hat{A}_1, \dots, \hat{A}_{n-1}$, on $\widehat{\Phi}(u_n)$ and u_n.

Now let us suppose that M_l is only proportional to the identity. This means that all eigenvalues $\lambda_1^{(l)}, \dots \lambda_m^{(l)}$ of A_l are resonant. Since their sum is zero, the only possibility is $M_l = \exp\left(\frac{2\pi i s}{m}\right) \mathbb{1}$ for some $s = 1, \dots, m-1$. To transform this matrix to the identity we use iterations of the symmetries (1.13), (1.14), to map our solution $A_1, \dots, A_n$ to a solution $\hat{A}_1, \dots, \hat{A}_n$ having $M_n = \mathbb{1}$. Since these symmetries are birational, $A_1, \dots, A_n$ are rational functions of $\hat{A}_1, \dots, \hat{A}_n$. Then we can apply the above procedure to kill $\hat{A}_n$.

Remark 3.3. Observe that in Lemma 3.2, for $M_l = \exp\left(\frac{2\pi}{m}\right) \mathbb{1}$, the new solution $\tilde{A}_1, \dots, \tilde{A}_{l-1}, \tilde{A}_{l+1}, \dots, \tilde{A}_n$ has monodromy matrices $M_1, \dots, M_{l-1}, M_{l+1}, \dots, M_n$, and a new monodromy matrix at infinity $\exp\left(-\frac{2\pi}{m}\right) M_\infty$.

Lemma 3.4. *Let $(A_1, \dots A_n)$ be a solution of the Schlesinger equations with M_∞ proportional to the identity $\mathbb{1}$, say $M_\infty = \exp\left(\frac{2\pi i}{m}\right)\mathbb{1}$. Suppose that M_n is not proportional to the identity, then there exists a solution $\tilde{A}_1, \dots \tilde{A}_{n-1}$ of the Schlesinger equations with monodromy matrices*

$$\mathcal{C}_n M_1 \mathcal{C}_n^{-1}, \dots, \mathcal{C}_n M_{n-1} \mathcal{C}_n^{-1}, \tag{3.6}$$

and $\widetilde{M}_\infty = \mathcal{C}_n \exp\left(-\frac{2\pi i}{m}\right) M_n \mathcal{C}_n^{-1}$, $\mathcal{C}_n$ being the connection matrix of M_n. The given solution $A_1, \dots A_n$ depends rationally on $\tilde{A}_1, \dots, \tilde{A}_{n-1}$, $\widetilde{\Phi}(u_n)$ and on u_n.

We perform a symmetry (1.14) (or a conformal transformation), in order to apply Lemma 3.2 to the case M_1 proportional to the identity.

End of the proof of Theorem 1.5. Suppose that $A_1, \dots, A_n$ is a solution of the Schlesinger equations such that the collection of its monodromy matrices $M_1, \dots, M_n$, M_∞ is l-smaller. If none of the monodromy matrices being proportional to the identity is equal to M_∞, we can simply conclude by l iterations of Lemma 3.2. If M_∞ is proportional to the identity, first we apply Lemma 3.4, then we iterate Lemma 3.2 $l-1$ times. This concludes the proof of Theorem 1.5. □

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On the Riemann–Hilbert correspondence for generalized Knizhnik–Zamolodchikov equations for different root systems

*Valentina A. Golubeva**

Pedagogical Institute of Kolomna
Faculty of Physics and Mathematics, Chair of Calculus
Zelyonaya street 30, 140411 Kolomnna, Moscow region, Russia
e-mail: golub@viniti.msk.su

Abstract. The paper is devoted to the restricted Riemann–Hilbert problem in a class of generalized Knizhnik–Zamolodchikov equations. The Knizhnik–Zamolodchikov equation has singularities on the set of reflection hyperplanes of the root system A_{n-1}. I. Cherednik proposed to consider also similar systems of equations, associated to the other root systems and gave examples of corresponding physical models. The Cherednik systems now are called the generalized Knizhnik–Zamolodchikov equations. In this class of equations the Riemann–Hilbert problem consists in investigating the correspondence between the equations and the representations of the fundamental group of the complement of the singular locus of the equation. For the case of the root system A_n this correspondence was investigated by V. Drinfeld and T. Kohno. In the paper an exposition of the results on the generalization of the Drinfeld–Kohno theory for the root system B_n obtained in collaboration with V. P. Lexin is given. Some principal elements of the proof of the main theorems are discussed. Besides, the equations of the Knizhnik–Zamolodchikov type for the other root systems and some equations close to them are touched.

Contents

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1 Introduction

The paper is devoted to one of several variants of the multidimensional Riemann–Hilbert problem. The first statements of this problem in the second half of the xxth century were given by I. M. Gelfand, T. Regge, O. S. Parasyuk and P. Deligne. The statement of the first three authors (published by T. Regge [55]) consists in proving the existence of some system of partial differential equations of generalized hypergeometric type which are satisfied by multivalued analytic functions with ramification on reducible algebraic varieties in $\mathbb{C}^n$ or in $\mathbb{CP}^n$, for example, as Landau varieties for Feynman integrals or the union of lines in $\mathbb{CP}^2$ for hypergeometric functions of two variables. Such problems became actual after the success in the investigation of analytical properties of the mentioned objects, especially Feynman integrals (see [22] and references). The first results along this line [22], [23], i.e. the derivation of the systems of equations of generalized Fuchsian type for the above-mentioned objects, permit to state the close relation of the multidimensional inverse problem with the corresponding one-dimensional Riemann–Hilbert problem in the theory of Fuchsian equations and equations with regular singularities. Regge's statement of the problem was modified and became very similar to the classical statement of Riemann–Hilbert problem [22]. Approximately at the same time as Regge's paper was published, R. Gérard [21] and P. Deligne [13] stated the multidimensional Riemann–Hilbert problem in the following terms: let $L = \bigcup L_i$ be a reducible algebraic variety in $\mathbb{CP}^n$ and

$$\rho\colon \pi_1(\mathbb{CP}^n \setminus L, x_0) \longrightarrow \mathrm{GL}(n, \mathbb{C})$$

a representation of the fundamental group of the complement to L. Does there exist a system of equations of the generalized Fuchsian type with monodromy equivalent to the given representation ρ?

It is necessary to note that, at that time, the one-dimensional Riemann–Hilbert problem was considered to be solved. But after the first investigations of the problem in the multidimensional case it became clear that the valuations used for solving local multidimensional Riemann–Hilbert problem [1], [4] have to play a principal role in the global solution of the one-dimensional Riemann–Hilbert problem. This fact was not taken into account in the classical papers of the beginning of the xxth century devoted to one-dimensional Riemann–Hilbert problem. But the real role of valuations became evident only after the Dekkers paper [12]. Realizing Dekkers's idea in the general case, A. Bolibrukh obtained an answer to Hilbert's question for the one-dimensional case [2], [3]. It became clear that the right statement of the Riemann–Hilbert problem has to contain the detailed characterization of a given representation of the fundamental group and some information concerning the class of equations which may be considered as solutions of the problem.

These observations became even more actual in the multidimensional case. After the first results in the multidimensional case connected with conditions of solvability of the multidimensional Riemann–Hilbert problem for the cases of the most simple fundamental groups (commutative or close to commutative, see [4], [22], [23], [24],

[56], [39], [36], [61] and references in the papers of A. Bolibrukh), it became clear that the statement of the inverse multidimensional Riemann–Hilbert problem has to contain a information as detailed as possible on a given representation and the characterization of a corresponding class of target differential equations since cherished hopes for a successful solution in general case are scanty. And the main attention has to be paid to the problem of investigating conditions for the solvability of the problem. Fortunately, at the end of the xxth century the physicists discovered some models of statistical mechanics and conformal field theory which were characterized by differential equations of generalized hypergeometric type [18], [40]. This was the class of equations which now is called Knizhnik–Zamolodchikov equations. These equations have logarithmic singularities on the set of hyperplanes which are the reflection hyperplanes of the root system A_{n-1}. Firstly, the Knizhnik–Zamolodchikov equations were introduced as equations for correlation function of Wess–Zumino–Witten model. I. Cherednik proposed to consider also similar systems of equations associated to the other root systems [5], [6] and presented examples of corresponding physical models [7], [8]. These systems are now called generalized Knizhnik–Zamolodchikov equations. For these equations the classical statement of this problem consisting in proving the existence of a system of differential equations with logarithmic singularities on some subvariety in CP^n (or an explicit constructing of such a system) if some representation of the complement to this subvariety is not good. Since the type of the system of differential equations is given, the problem in this case is to state the correspondence between some given representation of the complement to the singular locus of the equations of the considered class and the monodromy representation of the equation. Obviously, a variety of different root systems and given equations, as well as that of given representations of the complement to the singular locus of the equations, can be considered. Such a problem in the class of generalized Knizhnik–Zamolodchikov equations is called restricted Riemann–Hilbert problem [25]. Such problems are interesting for describing physical models with symmetries connected to the semisimple Lie algebras.

The first solution of the correspondence problem for the root system A_{n-1} – that is, for classical Knizhnik–Zamolodchikov equation (below we will use the notation KZ instead of Knizhnik–Zamolodchikov) – was given by V. Drinfeld and T. Kohno in terms of braided quasi-bialgebras and their one-parametric deformations [16], [17], [41], [42]. An excellent exposition of these results is given by Ch. Kassel [35].

Physically, this problem is related to the scattering process of elementary particles in quantum electrodynamics. A generalization of this is the scattering process with reflections [43]. For example, such are the Potts model [9], the Gaudin magnetic with reflecting boundary [33] or the Hubbard spin chain with open boundary conditions [31]. Such models are related with algebras of symmetries different from A_{n-1}. Some of these models are described by means of the KZ equations associated to the root system B_n. This root system has the following principal peculiarity: the roots under action of the Weyl group are divided into two orbits of roots of different length. As usually in the Lie algebras theory, the characterization of the symmetry properties of the model in this case is based on two-parametric deformations of the algebraic characteristics: one parameter is responsible for scattering processes of the particles, the other one corresponds to the characterization of the reflection processes.

In the paper an exposition of the results on the generalization of the Drinfeld–Kohno theory for the root system B_n obtained in collaboration with V.P. Lexin is given (see [26], [27], [28], [29], [47], [48]). We treat the following questions:

- the generalized KZ equation of the B_n type (one-parametric and two-parametric cases);
- geometric realization as symmetric braids of the Brieskorn braid group of the B_n type;
- the braided quasi-bialgebra associated with the KZ equation of the B_n type and the monodromy representation for this equation;
- the definition of the braided quasi-bialgebra of the B_n type on the multi-parametric deformation of the Drinfeld–Jimbo algebra and the representation of the Brieskorn braid group of the B_n type (its notation $B(B_n)$) in terms of the structural elements of the braided quasi-bialgebra of the B_n type;
- the axioms for the defining elements of the braided quasi-bialgebra of the B_n type.

Some principal elements of the proof of the main theorem are presented. Besides, the equations of the KZ type for the other root systems and equation close to them are considered.

2 Generalized Knizhnik–Zamolodchikov equation associated to the root system B_n

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra, $A = \|a_{ij}\|$ its Cartan matrix and $(d_1, \dots, d_r)$ squares of the root lengths with values 1, 2, 3. The construction of the Drinfeld–Kohno theory is based on the one-parametric deformations of the universal enveloping algebra $U(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$. The extension of this theory naturally could be based on one-parametric deformations as well as on multiparametric ones. The corresponding formal variables will be denoted by $h(d_i)$, $i = 1, 2$. For the simple Lie algebras A_n, D_n, E_6, E_7, E_8 we need one formal parameter, for algebras B_n, $(n \geq 2)$ and G_2 we will use one and two parameters, $U_h(\mathfrak{g})$ is a quantum universal enveloping algebra (Drinfeld–Jimbo algebra), where $h = (h_1, h_2)$, are complex parameters,$(U(\mathfrak{g})[[h]], \mu, \Delta)$ is a trivial deformation of $U(\mathfrak{g})$, the cases $(h_1 = h_2)$ will also be considered. The Belavin–Drinfeld tensor $t \in \mathfrak{g} \otimes \mathfrak{g}$, is

$$t = \frac{1}{2}(\Delta(c) - 1 \otimes c - c \otimes 1)$$

where c is the Casimir element in $\mathfrak{g}$.

Let H_B be the union of the reflection hyperplanes of the Weyl group of the root system B_n in $\mathbb{C}^n$

$$H_{ij}^- = \{z_i - z_j = 0, 1 \le i < j \le n\},$$
$$H_{ij}^+ = \{z_i + z_j = 0, 1 \le i < j \le n\},$$
$$H_k^0 = \{z_k = 0, 1 \le k \le n\}.$$

and let Y_n be its complement in $\mathbb{C}^n$, $Y_n = \mathbb{C}^n \setminus H_B$. The fundamental group $\pi_1(Y_n, y_0)$, $y_0 \in Y_n$, is the generalized pure braid group denoted by $P(B_n)$.

We consider the generalized KZ(B) equation associated to the root system B_n

$$d\Psi(z) = \Omega_{B_n},$$

where

$$\Omega_{B_n}(h_1, h_2) = \left(\frac{h_1}{2\pi i} \sum_{1 \le i < j \le n} \left(\frac{t_{ij}^- d(z_i - z_j)}{z_i - z_j} + \frac{t_{ij}^+ d(z_i + z_j)}{z_i + z_j}\right) + \frac{h_2}{2\pi i} \sum_{i=1}^n \frac{t_i^0 dz_1}{z_i}\right)\Psi(z).$$

Since the explicit form of the coefficients of the equation is known only in one-parametric case (see [46]), we first consider the case $h_1 = h_2 = h$. The coefficients $t_{ij}^-, t_{ij}^+, t_i^0$ of the form $\Omega_{B_n}(h, h)$ are defined by the elements $t^- \in U(g)^{\otimes 2}$, $t^0 \in U(\mathfrak{g})$ and by the Weyl–Chevalley automorphism $\sigma_W : U(\mathfrak{g}) \to U(\mathfrak{g})$.

Let R be the set of root vectors of the Lie algebra $\mathfrak{g}$, and R_+ the set of positive roots. We denote by e_α, $\alpha \in R$, root vectors, and by h_j a basis of the Cartan subalgebra of $\mathfrak{g}$. Then the Weyl–Chevalley automorphism σ_W acts by the rule:

$$\sigma_W(e_\alpha) = e_{-\alpha}, \quad \sigma_W(h_i) = -h_i, \quad \sigma(1) = 1.$$

The Belavin–Drinfeld tensor t and the Leibman element t^0 have the following form:

$$t = \sum_{\alpha \in R_+} e_\alpha \otimes e_{-\alpha} + \sum_{i,j=1}^n g_{ij} h_i \otimes h_j,$$

and

$$t^0 = \sum_{\alpha \in R_+} (e_\alpha e_{-\alpha} + e_{-\alpha} e_\alpha)$$

(see [46]). Let further $t^+ = (\sigma_W \otimes 1)t$. Besides, by $t_{ij}^-, t_{ij}^+, t_i^0$ we denote the images of the elements t^-, t^+, t^0 under the natural inclusions

$$U(\mathfrak{g})^{\otimes 2} \longrightarrow U(\mathfrak{g})^{\otimes n}, \quad \text{or} \quad U(\mathfrak{g}) \longrightarrow U(\mathfrak{g})^{\otimes n}$$

on the i-th and the j-th factors for $t^\pm$ and on the i-th factor for t^0 respectively.

The coefficients of the KZ equation of the B_n type satisfy the relations

$$[t_{ik}^-, t_{ij}^- + t_{jk}^-] = 0, \quad [t_{ij}^+, t_{ik}^- + t_{jk}^+] = 0, \quad [t_{ij}^-, t_{ik}^+ + t_{jk}^+] = 0,$$

and the following relations are also fulfilled:

$$[t_{ij}^- + t_i^0 + t_j^0, t_{ij}^+] = 0, \quad [t_{ij}^+ + t_i^0 + t_j^0, t_{ij}^-] = 0, \quad [t_{ij}^\pm, t_{kl}^\pm] = 0, \quad [t_{ij}^\pm, t_l^0] = 0.$$

One can show that these relations are equivalent to the integrability condition $\Omega \wedge \Omega = 0$ for the 1-form

$$\Omega_{B_n}(h,h) = \frac{h}{2\pi i}\left(\sum_{i<j}\left(\frac{t_{ij}^- d(z_i - z_j)}{z_i - z_j} + \frac{t_{ij}^+ d(z_i + z_j)}{z_i + z_j}\right) + \sum_{i=1}^n \frac{t_i^0 dz_i}{z_i}\right).$$

We consider the algebraic symmetries of the KZ equation of the B_n type. The 1-form Ω_{B_n} is invariant with respect to the Weyl group $W_{B_n} = S_n \ltimes (\mathbb{Z}_2)^n$, where S_n is the permutation group acting on $U(\mathfrak{g})^{\otimes n}$ by transpositions of the tensor factors and the generators ε_i of $(\mathbb{Z}_2)^n$ act as the Weyl–Chevalley automorphism $\sigma_{W,i}$ on the i-th tensor factor of $U(\mathfrak{g})^{\otimes n}$.

We have

$$s \cdot t_{ij}^\pm = t^\pm_{(s^{-1}i)(s^{-1}j)}, \quad s \cdot t_i^0 = t^0_{s^{-1}(i)}, \quad s \in S_n.$$

And the group $\mathbb{Z}_2^n$ acts on the coefficients of the form Ω_{B_n} in the following manner. Let $\varepsilon_i = \pm 1$, $i = 1, \dots, n$ and $(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ be the generators of $\mathbb{Z}_2^n$. We have

$$\begin{aligned} \varepsilon_i t_{kl}^\pm &= t_{kl}^\mp, \quad \text{if } i = k \text{ or } l; \\ \varepsilon_i t_{kl}^\pm &= t_{kl}^\pm, \quad \text{if } i \neq k, \neq l; \\ \varepsilon_k t_i^0 &= t_i^0. \end{aligned}$$

The action of W_{B_n} on the coordinates of $\mathbb{C}^n$ is characterized by the following rules: if $s \in S_n$, we have

$$s(z_1, \dots, z_n) = (z_{s^{-1}(1)}, \dots, z_{s^{-1}(n)}), \quad \varepsilon_i(z_1, \dots, z_n) = (z_1, \dots, -z_i, \dots, z_n).$$

Now consider the two-parametric case, $h_1 \neq h_2$. The coefficients $t^- = \tau$, $t^+ = \mu \in U(\mathfrak{g})^{\otimes 2}$, $t^0 = \nu \in U(\mathfrak{g})$ of the 1-form $\Omega_{B_n}(h_1, h_2)$ satisfy the Frobenius integrability conditions $d\Omega_{B_n} = 0$ and $\Omega_{B_n} \wedge \Omega_{B_n} = 0$. These conditions are equivalent to the following commutation relations:

$$[\tau_{ij}, \tau_{ik} + \tau_{jk}] = 0, \quad [\tau_{ij}, \mu_{ik} + \mu_{jk}] = 0, \quad [\mu_{ik}, \tau_{ij} + \mu_{jk}] = 0, \quad i \neq j \neq k,$$

$$[\tau_{ij}, \mu_{ij}] = 0, \quad [\tau_{ij}, \mu_{kl}] = 0, \quad i \neq j \neq k \neq l, \qquad [\nu_i, \nu_j] = 0, \quad i \neq j,$$

$$\begin{gathered} [\mu_{ij}, \nu_i + \nu_j] = 0, \quad [\tau_{ij}, \nu_i + \nu_j] = 0, \quad [\tau_{ij} + \mu_{ij}, \nu_i] = 0, \\ [\tau_{ij}, \nu_k] = 0, \quad [\mu_{ij}, \nu_k] = 0, \quad i \neq j \neq k. \end{gathered}$$

For $h_1 \neq h_2$ explicit formulae for t^+, t^-, t^0 are not known.[1] It is possible, that the problem can be solved in terms of higher Casimir elements and using the results of E. Vinberg's paper on the commutative subalgebras of universal enveloping algebras [60].

[1] **Added in proof.** Now some solutions of these commutation relations in terms of spin operators are obtained by the author.

3 Braided quasi-bialgebras of the Coxeter type B_n

In this section we define the main notions necessary for the generalization of the Drinfeld–Kohno theorem for the case of the root system B_n. As in the case of the root system A_{n-1}, the B_n theory is based on braided quasi-bialgebras. Recall shortly the geometric interpretation of the braid group of the B_n type. Here it is denoted $B(B_n)$. It should be remarked that although the fundamental group of the complement to singular locus of the KZ equation (associated to some root system) is a pure braid group, the monodromy representations and holonomy could be considered for the ordinary (not pure) braid group. This fact follows from the symmetry properties of the KZ equations.

We recall that the Brieskorn braid group $B(B_n)$ has generators $\sigma_1, \dots, \sigma_n, \tau$ that are connected by the relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i, \quad \text{if } |i-j| \geq 2,$$
$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1},$$
$$(\tau \sigma_1)^2 = (\sigma_1 \tau)^2, \quad \tau \sigma_i = \sigma_i \tau \quad \text{for } i \geq 2.$$

The geometric realization of the braid group $B(B_n)$ is given in the papers [14], [15], [44], [27], [30], [59], [54]. We recall the geometric realization of the braid group $B(B_n)$ as braids symmetric with respect to some axis. The construction of the generators of $B(B_n)$ is the following: first, consider the generators of the Artin braid group with $2n$ strings (classical notation B_{2n}) and choose in it the braids symmetric with respect to the axis passing through the origin in the vertical direction). The generators σ_i and τ are presented in Figure 1 and Figure 2. It is easy to verify that all necessary relations are fulfilled.

$\sigma_i =$

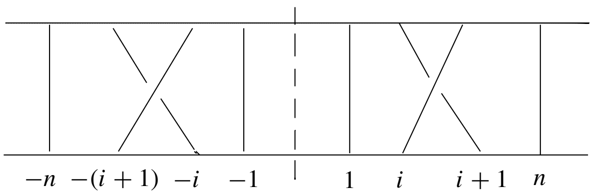


Figure 1

$\tau =$

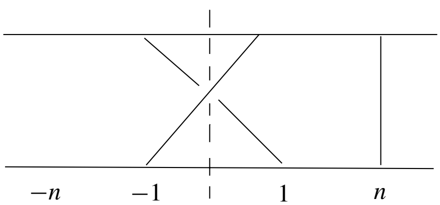


Figure 2

3.1 Braided quasi-bialgebra structures and twists

A braided quasi-bialgebra of the B_n type is some quasi-cocommutative and quasi-coassociative bialgebra of the B_n type with the following set of structural elements

$$\big(B, \mu, \Delta, \eta, \varepsilon, R_A, \Phi_A, R_B, \Phi_B\big),$$

where μ is the multiplication, Δ is the comultiplication, η a unit, ε a counit, $R_A \in B^{\otimes 2}$ is the R-matrix of the A-type and $R_B \in B$ is the R-matrix of the B-type [43], [26], [11], [32] and $\Phi_A \in B^{\otimes 3}$, $\Phi_B \in B^{\otimes 2}$ are the associators of A and B type respectively (see [19], [11], [47], [48]).

The following axioms for the structural elements hold:

(1) $\Delta_1 R_A = \Phi_A^{312} R_A^{13} (\Phi_A^{132})^{-1} R_A^{23} \Phi_A^{123}$,
$\Delta_2 R_A = (\Phi_A^{231})^{-1} R_A^{13} \Phi_A^{213} R_A^{12} (\Phi_A^{132})^{-1}$;

(2) $\Delta(R_B) = \Phi_B^{-1}(R_B \otimes I)\Phi_B R_A^{12} \Phi_B^{-1}(I \otimes R_B)\Phi_B$;

(3) $\Delta_3(\Phi_A)\Delta_1(\Phi_A) = (I \otimes \Phi_A)\Delta_2(\Phi_A)(\Phi_A \otimes I)$;

(4) $\Delta_1(\Phi_B) = \Phi_A^{-1}(\Phi_B \otimes I)\Phi_A$,
$\Delta_2 \Phi_B = \Phi_B \otimes 1$;

(5) $\varepsilon_1(R_A) = \varepsilon_2(R_A) = I \otimes I$;

(6) $\varepsilon_1(\Phi_A) = \varepsilon_2(\Phi_A) = \varepsilon_3(\Phi_A) = I \otimes I \otimes I$,
$\Delta_i = \Delta \otimes 1, 1 \otimes \Delta, \Delta \otimes I \otimes I, I \otimes \Delta \otimes I, I \otimes I \otimes \Delta$,
$\varepsilon_i = \varepsilon \otimes I, I \otimes \varepsilon, \varepsilon \otimes I \otimes I, I \otimes \varepsilon \otimes I, I \otimes I \otimes \varepsilon$;

(7) $\varepsilon(R_B) = I$, $(\varepsilon \otimes I)\Phi_B = (I \otimes \varepsilon)\Phi_B = I \otimes I$.

These axioms characterize the holonomy properties of a KZ equation of the B_n type that follow from permutability of the regularized holonomy with operation of symmetrical infinitesimal doubling of strings with free ends [48]. The geometric interpretation of some of the axioms is given in Figure 3.

In general a braided quasi-bialgebra structure of the B_n type on an algebra B with involution σ_B permits to define a representation of the Brieskorn braid group $B(B_n)$:

$$\rho_B\colon B(B_n) \longrightarrow B^{\otimes n}.$$

We have

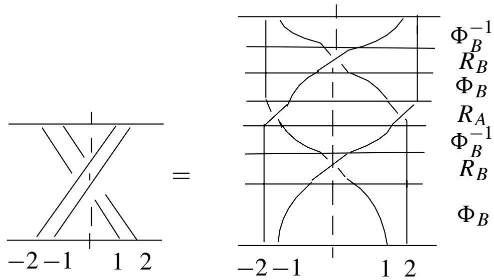
$$\begin{aligned} \rho_{\mathcal{B}}(\sigma_1) &= R_A s_{12}, \\ \rho_{\mathcal{B}}(\sigma_i) &= \Phi_A^{-1}(R_A)_{i,i+1} s_{i,i+1} \Phi_A, \\ \rho_{\mathcal{B}}(\tau) &= \Phi_B^{-1}(R_B)_1 \sigma_{W,1} \Phi_B. \end{aligned} \tag{1}$$

where $s_{i,i+1}$ is the transposition operator of the i-th and the j-th factors in $U(\mathfrak{g})^{\otimes n}$, $\sigma_{W,1}$ is the Chevalley–Weyl automorphism.

We define now a braided quasi-bialgebra structure connected with the KZ equation of the B_n type. It is characterized by the following data:

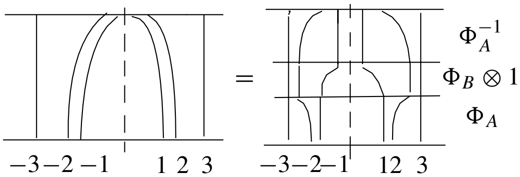
$$\mathcal{B}_{\mathrm{KZ}} = \big(U(\mathfrak{g})[[h_1, h_2]], \mu, \Delta, \eta, \varepsilon, R_A^{\mathrm{KZ}}, R_B^{\mathrm{KZ}}, \Phi_A^{\mathrm{KZ}}, \Phi_B^{\mathrm{KZ}}\big)$$

$\Delta R_B =$



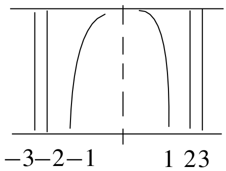
$$\Delta R_B = \Phi_B^{-1} \cdot (R_B \otimes 1) \cdot \Phi_B \cdot R_A \cdot \Phi_B^{-1} \cdot (1 \otimes R_B) \cdot \Phi_B$$

$\Delta_1(\Phi_B) =$



$$\Delta_1 \Phi_B = \Phi_A^{-1} \cdot (\Phi_B \otimes 1) \cdot \Phi_A$$

$\Delta_2(\Phi_B) =$



$= \Phi_B \otimes 1$

Figure 3

where

$$\begin{aligned}
R_A^{\mathrm{KZ}} &= e^{h_1 t^-/2} \in (U(\mathfrak{g})[[h_1, h_2]])^{\otimes 2},\\
R_B^{\mathrm{KZ}} &= e^{h_2 t^0/2} \in U(\mathfrak{g})[[h_1, h_2]],\\
\Phi_A^{\mathrm{KZ}} &\in (U(\mathfrak{g})[[h_1, h_2]])^{\otimes 3},\\
\Phi_B^{\mathrm{KZ}} &\in (U(\mathfrak{g})[[h_1, h_2]])^{\otimes 2}.
\end{aligned}$$

The comultiplication Δ is such that it is in a trivial deformation of $U(\mathfrak{g})$.

The formulas for the monodromy of the KZ equation of the B_n type defined on the generators σ_i, $i = 1, \dots, n-1$, and τ have the following form (see [47]):

$$\begin{aligned}
\rho_{\mathrm{KZ}}^{B_n}(\sigma_1) &= s_{12} e^{h_1 t_{12}^-/2},\\
\rho_{\mathrm{KZ}}^{B_n}(\sigma_i) &= ((\Phi_A^{\mathrm{KZ}})^{(i)})^{-1} s_{i i+1} e^{h_1 t_{i i+1}^-/2} (\Phi_A^{\mathrm{KZ}})^{(i)},\\
\rho_{\mathrm{KZ}}^{B_n}(\tau) &= (\Phi_B^{\mathrm{KZ}})^{-1} \sigma_{W,1} e^{h_2 t_1^0/2} \Phi_B^{\mathrm{KZ}},
\end{aligned} \tag{2}$$

where $\Phi_B^{\mathrm{KZ}}\big(\frac{h_1}{2\pi i}t_{12}^{+}, \frac{h_1}{2\pi i}t_{12}^{-}, \frac{h_2}{2\pi i}t_1^{0}\big)$ is the Drinfeld associator of the B_n type, $\sigma_{W,1}$ is the Chevalley–Weyl automorphism and $s_{i,i+1}$ is the transposition operator of the i-th and j-th factors in $U(\mathfrak{g})^{\otimes n}$. The $\Phi^{(i)}$ denotes $\Delta^{(i-1)}\Phi \otimes 1^{\otimes(n-i-1)}$, $\Delta^{(i+1)}\colon B^3 \to B^{\otimes(i+1)}$. The subscript "1" denotes the inclusion into the first factor.

Now we define a twisting of a quasi-bialgebra of the B_n type on $U(\mathfrak{g})[[h_1, h_2]]$. It is defined by means of two series $F_A \in U(\mathfrak{g})[[h_1, h_2]]^{\otimes 2}$ and $F_B \in U(\mathfrak{g})[[h_1, h_2]]$. The twisted structure is obtained in the following way:

$$\begin{gathered} \widetilde{R}_B = F_B R_B (F_B)^{-1}, \quad \widetilde{R}_A = F_A R_A F_A^{-1}, \\ \widetilde{\Phi}_B = \Delta F_B \Phi_B (\Delta F_B)^{-1}, \quad \widetilde{\Phi}_A = (1 \otimes F_A)\Delta_2 F_A \Phi_A \Delta_1 (F_A^{-1})(F_A^{-1} \otimes 1), \\ \widetilde{\Delta}(a) = F_A \Delta(a) F_A^{-1}, \quad \widetilde{\mu}(a \otimes b) = \mu(F_A(a \otimes b)F_A^{-1}). \end{gathered} \tag{3}$$

The following Proposition 1 holds.

Proposition 1. *This set of twisted objects of braided quasi-bialgebra structure satisfy Axioms* (1)–(7).

The proof is made by direct calculations. Here we consider only Axiom 2:

$$\Delta(R_B) = \Phi_B^{-1}(R_B \otimes I)\Phi_B R_A^{12} \Phi_B^{-1}(I \otimes R_B)\Phi_B.$$

We have to show that

$$\widetilde{\Delta}(\widetilde{R}_B) = \widetilde{\Phi}_B^{-1}(\widetilde{R}_B \otimes I)\widetilde{\Phi}_B \widetilde{R}_A^{12} \widetilde{\Phi}_B^{-1}(I \otimes \widetilde{R}_B)\widetilde{\Phi}_B.$$

After using the twist formulas (3), we obtain

$$\begin{aligned} (\widetilde{\Delta}\widetilde{R}_B) &= F_A(\Delta F_B)(\Delta R_B)\Delta(F_B^{-1})F_A^{-1} \\ &= F_A \Delta F_B \Phi_B^{-1}(R_B \otimes I)\Phi_B R_A \Phi_B^{-1}(I \otimes R_B)\Phi_B \Delta F_B^{-1} F_A^{-1}. \end{aligned}$$

We need to prove

$$\begin{aligned} & F_A \Delta F_B \Phi_B^{-1}(R_B \otimes I)\Phi_B R_A \Phi_B^{-1}(I \otimes R_B)\Phi_B \Delta F_B^{-1} F_A^{-1} \\ &\quad = \Delta F_B \Phi_B^{-1} \Delta F_B^{-1}(F_B R_B F_B^{-1} \otimes I) \\ &\qquad (\Delta F_B \Phi_B \Delta F_B^{-1})(F_A R_A F_A^{-1})(\Delta F_B \Phi_B^{-1} \Delta F_B^{-1}) \\ &\qquad (I \otimes F_B R_B F_B^{-1})(\Delta F_B \Phi_B \Delta F_B^{-1}). \end{aligned}$$

If we suppose that $\Delta F_B = F_A = F_B \otimes F_B$, we obtain the desired equality.

The proofs for the other axioms are similar to the case of the root system A_{n-1}.

Remarks. (1) It is easy to see that the reflection equation follows from Axiom 2.

(2) Original axioms and twists are given in B. Enriquez's paper [19].

3.2 Generalization of the Drinfeld–Kohno theorem

The Drinfeld–Kohno theorem for the root system A_{n-1} is the statement on the equivalence of two braided quasi-bialgebra structures. For the case the root system B_n we can define two quasi-bialgebra structures. One of them is

$$\mathcal{B}_h = \big(U_h(\mathfrak{g}), \mu_h, \Delta_h, \eta_h, \varepsilon_h, R_{A,h}, R_{B,h}, \Phi_{A,h} = 1 \otimes 1 \otimes 1, \Phi_{B,h} = 1 \otimes 1\big).$$

Any such bialgebra defines a representation of a braid group $B(B_n)$. In particular, we obtain the elements $\hat{R}_{A,h} = s_{12} R_{A,h}$ and $\hat{R}_{B,h} = \sigma_W R_{B,h}$ satisfying the braid relation of the type B_n

$$\hat{R}_{A,h}(\hat{R}_{B,h} \otimes 1)\hat{R}_{A,h}(1 \otimes \hat{R}_{B,h})) = (\hat{R}_{B,h} \otimes 1)\hat{R}_{A,h}(1 \otimes \hat{R}_{B,h}))\hat{R}_{A,h}.$$

The other quasi-bialgebra structure is

$$\mathcal{B}_{\mathrm{KZ}} = \big(U(\mathfrak{g})[[h]], \mu, \Delta, \eta, \varepsilon, R_A = e^{t^-/2}, R_B = e^{ht^0/2}, \Phi_A^{\mathrm{KZ}}, \Phi_B^{\mathrm{KZ}}\big).$$

The existence of the element R_B for $\mathfrak{g} = \mathfrak{sl}(2)$ is proved in [32] (see also [11]). The existence problems and explicit form for the associators for the B_n case are considered in [19]. It would be desirable to obtain explicit formulas for the R_B matrix for the case of the root system B_n such as formulas for the R_A matrix obtained in [56], [58], [38], [37].

Below we give two theorems generalizing the Drinfeld–Kohno results for the KZ equations associated with the root system B_n.

Theorem 1. *There exist two twisting series $F(h_1, h_2) = F_A, F_B$ such that the following isomorphism of the quasi-bialgebras holds:*

$$\big(\mathcal{B}_{\mathrm{KZ}}\big)_{F(h_1,h_2)} \cong \mathcal{B}_h.$$

The proof of the theorem is given in [28], [29] (in press), see some elements of the proof in the next section. The proof of the theorem consists of several steps, one of them is the proof of the isomorphism up to a twist of two quasi-bialgebra structures whose structural elements coincide with the trivial multiparametric deformation of $U(g)$, except for one associator Φ_A. Similar assertion holds for the associator Φ_B. The proof of the last fact is based on cyclic cohomology [10], [49]. The next step is the proof of the fact that the KZ bialgebraic structure is isomorphic up to a twist to the braided bialgebra structure on the Drinfeld–Jimbo algebra with universal R_A and R_B R-matrices. The reduction of the quasi-bialgebra structure $\mathcal{B}_{\mathrm{KZ}}$ to the braided bialgebra structure with trivial associators Φ_A and Φ_B is used.

Using the definition of the twist (3) we can show that two isomorphic quasi-bialgebras define equivalent representations. Since the explicit form of the coefficients of the KZ equation of the B_n type is given only in the one-parametric case, we give the statement of the following theorem only for this case (see [28]). It is a corollary of Theorem 1.

Theorem 2. *Let g be a simple Lie algebra of the B_n type, and let $t^- \in g\otimes g \subset U(g)^{\otimes 2}$, $t^+ \in g\otimes g \subset U(g)^{\otimes 2}$, $t^0 \in U(g)$ be elements satisfying commutation Frobenius relations. Then the monodromy representation of the KZ equation of the type B_n with coefficients defined in terms of the elements $t^{\pm}$, t^0*

$$\rho_{\mathrm{KZ}}\colon B(B_n) \longrightarrow (U(\mathfrak{g})[[h]])^{\otimes n}$$

satisfying

$$\rho^{B_n}_{\mathrm{KZ}}(\sigma_1) = s_{12}e^{h_1 t^-_{12}/2},$$
$$\rho^{B_n}_{\mathrm{KZ}}(\sigma_i) = ((\Phi^{\mathrm{KZ}}_A)^{(i)})^{-1} s_{i\,i+1} e^{h_1 t^-_{ii+1}/2} (\Phi^{\mathrm{KZ}}_A)^{(i)},$$
$$\rho^{B_n}_{\mathrm{KZ}}(\tau) = (\Phi^{\mathrm{KZ}}_B)^{-1}\sigma_{W,1} e^{h_2 t^0_1/2}\Phi^{\mathrm{KZ}}_B$$

according to formulas (2)*, is equivalent to the representation of the braid group of the B_n type defined in terms of the braided quasi-bialgebra structure on the Drinfeld–Jimbo algebra with associator $\Phi_A = 1\otimes 1\otimes 1$ and $\Phi_B = 1\otimes 1$,*

$$\rho_B\colon B(B_n) \longrightarrow B^{\otimes n}$$

with, according to formulas (1)*,*

$$\rho_{\mathcal{B}}(\sigma_1) = R_A s_{12},$$
$$\rho_{\mathcal{B}}(\sigma_i) = \Phi_A^{-1}(R_A)_{i,i+1}s_{i,i+1}\Phi_A,$$
$$\rho_{\mathcal{B}}(\tau) = \Phi_B^{-1}(R_B)_1\sigma_{W,1}\Phi_B.$$

4 The rigidity theorems for the braided quasi-bialgebras of the Coxeter type B_n

The object of this section is to give some non trivial elements of the proof of the Theorems 1 and 2. We give a construction of some braided quasi-bialgebra structure which is equivalent to the structure $\mathcal{B}_{\mathrm{KZ}}$ and give the quantization of the initial Lie bialgebra of the B_n type. The method is based (similarly as in the one-parametric case) on the construction of some twisting transformation that permits to trivialize both associators of the $\mathcal{B}_{\mathrm{KZ}}$ structure.

The following two rigidity propositions holds.

Proposition 2. *The braided quasi-bialgebra structure $\mathcal{B}_{\mathrm{KZ}}$ of the B_n type is equivalent to a braided bialgebra structure $\widetilde{\mathcal{B}}_{\mathrm{KZ}}$ with trivial associators $\Phi_A = 1\otimes 1\otimes 1$, $\Phi_B = 1\otimes 1$.*

Proposition 3. *For the Drinfeld–Jimbo algebra $U(\mathfrak{g})_{h_1,h_2}(g)$ there exist universal elements $R_{h_1,h_2,A} \in U_{h_1,h_2}(g)^{\otimes 2}$ and $R_{h_1,h_2,B} \in U_{h_1,h_2}(g)$ such that the braided bialgebra structure $\mathcal{B}_{h_1,h_2} = (U_{h_1,h_2}(g), \mu, \Delta, R_{h_1,h_2,A}, R_{h_1,h_2,B})$ is isomorphic to the structure $\widetilde{\mathcal{B}}_{\mathrm{KZ}}$.*

This structure $\mathcal{B}_{h_1,h_2}$ gives a two-parametric quantization of the initial Lie bialgebra (g, δ_r) of B_n type.

The proof of these propositions is based on the construction of the series for the twisting elements trivializing the associators. At first, an isomorphism of the structures

$$\mathcal{B}_0 = (A, \mu, \Delta, R_A, R_B, \Phi_A, \Phi_B), \quad \mathcal{B}_1 = (A, \mu, \Delta, R_A, R_B, \Phi'_A, \Phi_B)$$

has to be proved. Then an analogous statement on the isomorphism of the structures

$$\mathcal{B}_2 = (A, \mu, \Delta, R_A, R_B, \Phi_A, \Phi_B), \quad \mathcal{B}_3 = (A, \mu, \Delta, R_A, R_B, \Phi_A, \Phi'_B)$$

has to be verified.

The proof of the trivialization of the associator of the A_n type is very similar to the proof of the analogous assertion for the one-parametric case and we do not give it here (see [28] and [29]). The procedure of the trivialization of the associator of the B_n type is not so trivial since it is based on an additional assumption on the obstruction cocycles. It is not difficult to verify that after proving the isomorphisms $\mathcal{B}_0 \simeq \mathcal{B}_1$ and $\mathcal{B}_2 \simeq \mathcal{B}_3$, we obtain the isomorphism

$$\mathcal{B} = (A, \mu, \Delta, R_A, R_B, \Phi_A, \Phi_B) \simeq \mathcal{B}' = (A, \mu, \Delta, R_A, R_B, \Phi'_A, \Phi'_B).$$

for $A = U(g)[[h_1, h_2]]^{\otimes 2}$ that implies the Proposition 2.

4.1 Construction of the twist trivializing the associator Φ_B

In this subsection we will consider the most difficult point of the proof of Proposition 5. It is based on the following lemma and two propositions given below.

Lemma. *Assume that B_2 and B_3 are two braided quasi-bialgebra structures for $A = U(g)[[h_1, h_2]]$ that differ only by the associators of B type, that is $\Phi_B \neq \Phi'_B$. Furthermore, suppose that Φ_B and Φ'_B are symmetrical tensors in $(U(g)[[h_1, h_2]])^{\otimes 2}$. Then there exists an invertible element $F_B \in U(g)[[h_1, h_2]]$ such that the following equality holds:*

$$\Phi'_B = \Delta F_B \Phi_B (\Delta F_B)^{-1}.$$

This means that the structures $\mathcal{B}_2$ and $\mathcal{B}_3$ are equivalent with respect to the twisting by means of the element F_B.

Proof of the lemma. Introduce the difference $\Psi_B = \Phi_B - \Phi'_B$ and suppose that we have an expansion of the form

$$\Psi_B = \sum_k \Psi_{B,k} h_1^{k_1} h_2^{k_2} + \cdots .$$

Firstly, as in [28] (see also [35]), we note that the first non-zero terms of the expansion satisfy the cocycle relation of some cohomology group of $U(g)$. Then we prove that these cocycles are cohomologous to zero. The cochains whose coboundaries give the cocycles define the terms of the corresponding addenda of the corresponding degrees of element F_B. □

The conditions that characterize the expansion of Ψ_B will be found from the axioms of Section 3. We use only Axioms 2 and 4 that contain Φ_B. For realizing this program, we have the following propositions.

Proposition 4. *Axiom* 2 *does not give relations for the first non-zero term of the expansion of* Φ_B *on the degrees of* h_1, h_2.

The proof of the proposition is very similar to the proof of the fact that Axiom 1 does not impose restrictions on the terms of the expansion of $\Psi_A = \Phi_A - \Phi'_A$ (see [28] and [29]).

For the statement of the following proposition we investigate the restrictions on the elements of Φ_B that follow from Axiom 4. We consider two braided quasi-bialgebras $\mathcal{B}_2$ and $\mathcal{B}_3$.

Introduce the notation: $h^{\bar{n}} = h_1^{n_2} h_1^{n_2}$.

Proposition 5. *For the first non-zero coefficient of the expansion* $\Psi_B = \Psi_{B,\bar{n}} h^{\bar{n}} + \cdots$ ($\bar{n} = (n_1, n_2)$*, using the lexicographical order*) *the following relations are fulfilled:* $\Delta_1 \Psi_{B,\bar{n}} = \Psi_{B,\bar{n}} \otimes 1$, $\Delta_2 \Psi_{B,\bar{n}} = \Psi_{B,\bar{n}} \otimes 1$.

From the first part of Axiom 4 it follows that in the two braided quasi-bialgebras $\mathcal{B}_2$ and $\mathcal{B}_3$ the following equations have to be fulfilled:

$$\Delta_1(\Phi_B) = {\Phi_A}^{-1} \cdot (\Phi_B \otimes I) \cdot \Phi_A, \quad \Delta_1(\Phi'_B) = \Phi_A^{-1} \cdot (\Phi'_B \otimes I) \cdot \Phi_A,$$

Subtracting these equations one from another we obtain

$$\Delta_1(\Phi_B - \Phi'_B) = \Phi_A^{-1} \cdot ((\Phi_B - \Phi'_B) \otimes I) \cdot \Phi_A,$$

that is $\Delta_1 \Psi_B = \Phi_A^{-1} (\Psi_B \otimes I) \Phi_A$. For the first non-zero addendum of $\Psi_{B,n}$ of the degrees of h_1, h_2 we have $\Delta_1 \Psi_{B,\bar{n}} = \Psi_{B,\bar{n}} \otimes 1$.

From the second part of Axiom 4 we obtain $\Delta_2 \Psi_{B,\bar{n}} = \Psi_{B,\bar{n}} \otimes 1$. Let $\Psi_{B,n}$ have the following form (we suppose $\bar{n} = (n_1, n_2)$, $|\bar{n}| = n_1 + n_2 = n$): $\Psi_{B,n} = \sum_{\bar{n}, |\bar{n}| = n} \Psi_{B,\bar{n}} h^{\bar{n}}$.

Consider the homology of the coalgebra $U(g)$ with coefficients in the $U(g)$-module $\mathbb{C}$. The corresponding complex has the form:

$$C^n(U(g)) = U(g)^{\otimes n}$$

with differential $d\colon C^n(U(g)) \to C^{(n+1)}(U(g))$ given by the formula $d = \sum_{i=0}^{n+1} \Delta_i$ where $\Delta^0(a) = 1 \otimes a$,

$$\Delta_i(a_1 \otimes \cdots \otimes a_n) = a_1 \otimes \cdots \otimes \Delta a_i \otimes \cdots \otimes a_n$$

for $1 \le i \le n$, $\Delta_{n+1}(a) = a \otimes 1$, $a \in C^n(U(g))$.

Consider now the cyclic homology of the coalgebra $U(g)$, that is the cohomology of the complex $CC^n(U(g)) = C^n(U(g))/(1-t)C^n(g))$ where $t(a_1 \otimes \dots \otimes a_n) = a_n \otimes a_1 \otimes \dots \otimes a_{n-1}$. Since by condition of the lemma Φ_B and Φ'_B are symmetrical elements, then $\Psi_{B,\bar{n}}$ are symmetric elements in $C^2(U(g))$. Therefore we have $\Psi_{B,\bar{n}} \in CC^2U(g))$, for all $\bar{n} \neq (0,0) \in \mathbb{Z} \oplus \mathbb{Z}$. Also, for $\Psi_{B,\bar{n}} \in CC^2(U(g))$ we have

$$d\Psi_{B,\bar{n}} = 1 \otimes \Psi_{B,\bar{n}} - \Psi_{B,\bar{n}} \otimes 1 = -(1-t)\Psi_{B,\bar{n}} \otimes 1 = 0 \in CC^3(U(g))$$

for all $\bar{n} \neq (0,0) \in \mathbb{Z} \oplus \mathbb{Z}$. Thus, $\Psi_{B,\bar{n}}$ are 2-cocycles in the cyclic cohomology. Consequently, under the symmetry condition for Φ_B we obtain the cocycles $\Psi_{B,\bar{n}}$ as elements of the cohomology group $HC^2(U(g))$

Since for the simple Lie algebra we have $H^1(g,\mathbb{C}) = H^2(g,\mathbb{C}) = 0$, from the results of the A. Connes and H. Moscovici [10] on cyclic cohomology, it follows that $HC^2(U(g)) = 0$. We obtain that $\Psi_{B,\bar{n}}$ is a coboundary, that is $dF_{B,\bar{n}} = \Psi_{B,\bar{n}}$.

In that case the twisting

$$F_B^{(n)} = 1 + \sum_{\bar{n},|\bar{n}|=n} F_{B,\bar{n}} h^{\bar{n}}$$

cancels the terms in the difference $\Psi_B = \Phi_B - \Phi'_B$. The same induction arguments and the consideration of the multiple twisting

$$F_B = \Pi_{n=1}^{\infty} F_B^{(n)}$$

implies the existence of the element satisfying the equality

$$\Delta F_B \Phi'_B (\Delta F_B)^{-1} = \Phi_B.$$

This assertion finishes the proof.

5 Equivalence problem for the other root systems

At the present time the situation with the investigation of the equivalence problem for the other root systems (C, D, G_2, F_4, E-series) is similar to the situation that prevailed for the root systems A_{n-1} and B_{n-1} twenty years ago, but it is even worse. Although the corresponding braid groups and many of their representations are known, and in general the form of the associated generalized KZ equations was conjectured by Cherednik, all this is not sufficient in order to state and to solve the equivalence problem for the monodromy representation. I shall touch some results that may help in the solution of the restricted Riemann–Hilbert problem for these root systems.

At first, consider the equations of the KZ type for the root system G_2. The first simple equations were written in the paper [62]. The representations of the Lie algebra G_2 and of the corresponding braid group were considered, for example, in the papers [52], [51], [34], [50].

At the present time some other forms of the KZ equations (different from the equation in [62]) associated to the root system G_2 are known, the so-called tensor form. Let $\mathfrak{g}$ be Lie algebra of type G_2. The corresponding system of positive roots in $\mathbb{R}^3$ in a canonical basis is $r_1 = \varepsilon_1 - \varepsilon_2$, $r_2 = \varepsilon_2 - \varepsilon_3$, $r_3 = \varepsilon_3 - \varepsilon_1$, $r_4 = r_1 - r_2$, $r_5 = r_2 - r_3$, $r_6 = r_3 - r_1$. We denote the linear functions defining the reflection hyperplanes of this root system by $P_i(x_1, x_2, x_3)$, $i = 1, \dots, 6$, and the equations of these hyperplanes then are $P_i = 0$. Now we can write one of possible variants of the corresponding equation of KZ type. One of the form is the following: $df = \Omega f$, where

$$\Omega = \frac{h}{2\pi i} \sum_{i=1}^{6} \tau_i \frac{dP_i}{P_i}.$$

Here $\tau_1 = t_{12}$, $\tau_2 = t_{23}$, $\tau_3 = t_{31}$, $\tau_4 = t_{12} - t_{23}$, $\tau_5 = t_{23} - t_{31}$, $\tau_6 = t_{31} - t_{12}$, t_{ij} is Belavin–Drinfeld tensor in $U(\mathfrak{g}) \otimes U(\mathfrak{g})$. If e_α is the Weyl root system and h_1, h_2 is a basis of the Cartan subalgebra over $\mathbb{C}$, we have

$$t = \sum_{\alpha \neq 0} e_\alpha \otimes e_{-\alpha} + \sum_{i,j=1,2} g_{ij} h_i h_j$$

where g_{ij} are elements of the inverse matrix to the matrix $H = \|(h_i, h_j)\|$. This tensor satisfies the commutation relations of Section 2. Further, for the construction of the matrix-tensor realization of the KZ equation we can use one of the matrix representations of the Lie algebra, for example, in the Gelfand–Zeitlin basis. The geometric realization of the corresponding braid group of type G_2 is known. Finding the pairs of the bialgebraic structures and proving their equivalence is an interesting problem that has to be solved. Evidently, the other forms of the KZ equations of G_2 type are possible, in particular, connected with representations considered in cited above papers. The corresponding equations may find applications in statistical mechanics and in the theory of elementary particles.

A similar problem can be stated for the root system F_4. Some appropriate paper is [52]. In both cases the paper [20] may appear useful. The cases of KZ equations for the root system of the E series and the super KZ equations will be considered in future publications.

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Monodromy groups of regular systems on the Riemann sphere

Vladimir Petrov Kostov

Laboratoire de Mathématiques
Université de Nice
Parc Valrose, 06108 Nice Cedex 2, France
email: `kostov@math.unice.fr`

Abstract. We consider the Riemann–Hilbert problem (or Hilbert's 21st problem) for Fuchsian linear systems of ordinary differential equations on the Riemann sphere: Prove that for any prescribed monodromy group and poles there exists a Fuchsian system with this monodromy group and with these poles. In this setting the problem has been given a negative answer due to A. A. Bolibrukh. We give sufficient conditions for the realizability of a monodromy group by a Fuchsian system, e.g. if one of the monodromy operators in its Jordan normal form has at most one block of size 2 the rest being of size 1, or if the monodromy group is irreducible (the proof of this result has been obtained simultaneously and independently of the one of Bolibrukh). We discuss invariants of matrix groups considered as monodromy groups. We also give the codimension in the space of couples (monodromy group, poles) of the set for which the answer to the Riemann–Hilbert problem is negative, and we describe the couples on which this codimension is attained.

Contents

1 Introduction and plan of the paper

In the present paper we consider various problems connected with linear *regular* and in particular *Fuchsian* systems of ordinary meromorphic differential equations on the Riemann sphere. "Regular" means that the growth rate of the solutions restricted to sectors with vertices at the poles of the system is *moderate* (or *polynomial*) when the poles are approached, i.e. the norm of the solution is bounded by some power of the distance to the given pole. "Fuchsian" means that the poles are of first order.

The only analytic invariant of a regular system is its *monodromy group*, see Subsection 2.2. A *monodromy operator* is a linear operator mapping the solution space onto itself, the image of a solution being its analytic continuation along some closed contour on the Riemann sphere bypassing the poles. At the poles the solutions, in general, have ramification points.

In Section 2 we consider the Riemann–Hilbert (or Hilbert's 21st) problem – whether any monodromy group can be obtained from a Fuchsian system with prescribed poles and no additional singularities. In such a generality the problem has a negative answer due to A. A. Bolibrukh, see [5]. In [5] and [1] one can find a detailed history of the problem as well. A generalization of the problem to systems with irregular singular points can be found in [13].

We give several sufficient conditions for a matrix group to be the monodromy group of a Fuchsian system with prescribed poles and no other singularities:

1) The group to be irreducible (regardless of the position of the poles), see Theorem 2.2. Its proof is algorithmic; a sketch of it appeared in [15]; another proof of the same sufficient condition is given in [6], which is more geometric and using the language of vector bundles. Both proofs were found independently and almost at the same time. The scheme of the algorithmic proof of the author is used in the present paper also to obtain other sufficient conditions.

2) One of the monodromy operators in its Jordan normal form to have at most one Jordan block 2×2, all the other Jordan blocks being 1×1 (again regardless of the position of the poles, see Corollary 5.6). This is an improvement of a similar result of Yu.S. Il'yashenko which requires one of the monodromy operators to be diagonalizable. One cannot improve further if one wants to give a sufficient condition

only in terms of one of the Jordan normal forms – there are counterexamples when there is one Jordan block of size 3 or two blocks of size 2.

Other sufficient conditions for the positive answer to the problem are formulated in Section 5. Some of them are better understood when the notion of a *monodromy stratification* of $(gl(n,\mathbb{C}))^p$ and of $(\mathrm{GL}(n,\mathbb{C}))^p$ is introduced, see Section 4; $(gl(n,\mathbb{C}))^p$ and $(\mathrm{GL}(n,\mathbb{C}))^p$ are regarded as the space of p-tuples $(A_1,\ldots,A_p)$ of residua of Fuchsian systems with $p+1$ poles (the last matrix-residuum equals $-A_1-\cdots-A_p$) and as the space of p-tuples $(M_1,\ldots,M_p)$ of the monodromy operators corresponding to encircling these poles once. The last monodromy operator equals $(M_1\ldots M_p)^{-1}$.

The definition of a stratum takes into account the Jordan normal forms of A_j (of M_j), $j=1,\ldots,p+1$, and the possible reducibility of the p-tuple (i.e. the existence of a common invariant proper subspace); in the case of $(\mathrm{GL}(n,\mathbb{C}))^p$ also some arithmetic properties of the eigenvalues have to be taken into account (whether the multiplicities of all eigenvalues of all matrices M_j have a non-trivial greatest common divisor or not). The definition is motivated by the negative answer to the Riemann–Hilbert problem (see [5]): for $n=3$ the negative answer to the problem is obtained for reducible monodromy groups whose monodromy operators $M_1,\ldots,M_{p+1}$ are conjugate to Jordan blocks 3×3.

In Section 3 a normal form of the centralizer of a reducible monodromy group in $\mathrm{GL}(n,\mathbb{C})$ is found in the generic case, i.e. when each matrix the conjugation with which leaves the group block upper-triangular is itself block upper-triangular. This normal form is based on a block decomposition defined by the dimensions of the invariant subspaces. Each block of a matrix from the centralizer is scalar or 0. The classification by the normal forms of the centralizer in the generic case involves no modules (which is not true for the non-generic case, see an example in Subsection 3.3).

In Section 6 we show that the codimension in $(\mathrm{GL}(n,\mathbb{C}))^p$ of the set of monodromy groups which cannot be obtained from Fuchsian systems with given poles equals $2p(n-1)$ for $p\geq 3$ and $n=3$ or $n\geq 7$; the result is true also for $p=2$ and n sufficiently large. We show in terms of the monodromy stratification for which strata there is an equality.

2 The Riemann–Hilbert problem

2.1 Regular and Fuchsian systems

A meromorphic linear system of ordinary differential equations on $\mathbb{C}P^1$ can be presented in the form

$$\dot{X}=A(t)X \tag{1}$$

where $A(t)$ is an $n\times n$ matrix-function meromorphic on $\mathbb{C}P^1$, "$\dot{}$" $=d/dt$. Denote its poles by $a_1,\ldots,a_{p+1}$, $p\geq 1$. We assume that the dependent variables X form also an $n\times n$-matrix. System (1) is called *Fuchsian* if all the poles of the matrix-function

$A(t)$ are of first order. A Fuchsian system can be written in the form

$$\dot{X} = \Big(\sum_{j=1}^{p+1} A_j/(t-a_j)\Big)X.$$

The matrices A_j are the *residua* of the system at the poles a_j; if there is no pole at ∞, then one has $A_1 + \cdots + A_{p+1} = 0$. System (1) is called *regular* at the pole a_j if

$$\|X(t-a_j)\| = O\big(|t-a_j|^{N_j}\big), \quad N_j \in \mathbb{R},\ j = 1, \dots, p+1.$$

Here $\|\cdot\|$ denotes an arbitrary norm in $gl(n,\mathbb{C})$ and we consider a restriction of the solution to a sector with vertex at a_j and of a sufficiently small radius, i.e. not containing other poles of $A(t)$. Every Fuchsian system is regular, see [29].

Two systems (1) with the same set of poles are called *equivalent* if there exists a meromorphic transformation (equivalence) on $\mathbb{C}P^1$

$$X \mapsto W(t)X \tag{2}$$

with $W \in \mathcal{O}(\mathbb{C}P^1\backslash\{a_1,\dots,a_{p+1}\}) \otimes gl(n,\mathbb{C})$ and $\det W(t) \neq 0$ for every t in $\mathbb{C}P^1\backslash\{a_1,\dots,a_{p+1}\}$ which brings the first system to the second one. A transformation (2) changes system (1) according to the rule

$$A(t) \to -W^{-1}(t)\dot{W}(t) + W^{-1}(t)A(t)W(t). \tag{3}$$

Remark 2.1. In recent times mathematicians tend to use more often the language of *meromorphic connections* than the one of linear systems of differential equations. Both languages are equivalent.

2.2 The monodromy group

The *monodromy group* of system (1) is defined as follows: fix a point $a \neq a_j$ for $j = 1, \dots, p+1$, fix a matrix $B \in \mathrm{GL}(n,\mathbb{C})$, and fix $p+1$ closed contours on $\mathbb{C}P^1$ beginning at the point a each of which contains inside exactly one of the poles a_j of the system (1) which it circumvents counterclockwise.

More exactly, we assume that the j-th contour γ_j consists of a segment $[a, a_j']$ where the point a_j' is close to a_j, of a circumference centered at a_j, of radius $|a_j' - a_j|$, run counterclockwise and containing inside no pole other than a_j, and of the segment $[a_j', a]$. We assume that for $j_1 \neq j_2$ one has $\gamma_{j_1} \cap \gamma_{j_2} = \{a\}$, and that one encounters successively the contours $\gamma_1, \dots, \gamma_{p+1}$ when one turns around a clockwise.

The *monodromy operator* corresponding to such a contour is the linear operator mapping the matrix B onto the value of the analytic continuation of the solution to system (1) which equals B for $t = a$ along the contour. The monodromy operators $M_1, \dots, M_p$ corresponding to $a_1, \dots, a_p$ generate the *monodromy group* of system (1). One has

$$M_{p+1} = (M_1 \dots M_p)^{-1}. \tag{4}$$

It is clear that

(1) the monodromy group is defined up to conjugacy due to the freedom to choose the point a and the matrix B;

(2) the monodromy groups of equivalent systems are the same.

2.3 The old and the new version of the problem

The Riemann–Hilbert problem (or Hilbert's 21st problem) is formulated as follows:

Prove that for any set of points $a_1, \dots, a_{p+1} \in \mathbb{C}P^1$ *and for any set of matrices* $M_1, \dots, M_p \in \mathrm{GL}(n, \mathbb{C})$ *there exists a Fuchsian linear system with poles at* $a_1, \dots, a_{p+1}$ *and only there for which the corresponding monodromy operators are* $M_1, \dots, M_p,\ \ M_{p+1} = (M_1 \dots M_p)^{-1}$.

The Riemann–Hilbert problem has a positive solution for $n = 2$ which is due to Dekkers, see [14]. For $n = 3$ the answer is negative, see [5]. It was proved by A. A. Bolibrukh, however, that for $n = 3$ the problem has a positive answer if we restrict ourselves to the class of systems with irreducible monodromy groups, see [5]. Later, the author (see [15] and [16]) and independently Bolibrukh (see [6]) proved this result for any n.

It has been believed for a long time that the problem has a positive solution for any $n \in \mathbb{N}$, after Plemelj in 1908 gave a wrong proof, see [26]. It nevertheless follows from his proof that if one of the monodromy operators of system (1) is diagonalizable, then system (1) is equivalent to a Fuchsian one, see [3]. It also follows that any finitely generated subgroup of $\mathrm{GL}(n, \mathbb{C})$ is the monodromy group of a regular system with prescribed poles which is Fuchsian at all the poles with the exception, possibly, of one which can be chosen among them arbitrarily.

It is reasonable to reformulate the Riemann–Hilbert problem as follows:

Find necessary and/or sufficient conditions for the monodromy operators $M_1, \dots,$ M_p *and the points* $a_1, \dots, a_{p+1}$ *so that there should exist a Fuchsian system with poles at and only at the given points and whose monodromy operators* M_j *should be the given ones.*

In this section we prove the following theorems.

Theorem 2.2. *Every finitely generated irreducible subgroup of* $\mathrm{GL}(n, \mathbb{C})$ *is the monodromy group of a Fuchsian system on* $\mathbb{C}P^1$ *with prescribed poles. The generators are assumed to correspond to the operators* $M_1, \dots, M_p$. *In other words, any regular system with irreducible monodromy group is equivalent to a Fuchsian one.*

We do not prove Theorem 2.2 directly, but the more detailed Theorem 2.12; its proof is algorithmic. Theorem 2.2 was proved in [16], a sketch of the proof is given in [15].

Remark 2.3. We do not allow any additional singularities (called *apparent* further in the text) the monodromy operators corresponding to which are equal to I. The reader will easily derive from the proof of Theorem 2.12 the fact that *any* monodromy group can be realized by a Fuchsian system with prescribed poles if we allow one apparent Fuchsian singularity at an arbitrary fixed point $a^0 \neq a_1, \ldots, a_{p+1}$.

Theorem 2.4. *For any given n and p (they have the same meaning as above) there exists a constant* $H(n, p) \leq (2n^2 - 3)(p + 1)$ *such that every finitely generated subgroup of* $\mathrm{GL}(n, \mathbb{C})$ *is the monodromy group of a regular system on* $\mathbb{C}P^1$ *with prescribed poles (and no apparent singularities) which is Fuchsian at all the poles with the exception, possibly, of one. The order of this pole is not greater than* $H(n, p)$.

A better estimation for $H(n, p)$ can be found in [1], p. 127. Theorem 2.4 is proved in Subsection 2.9. In its proof we use

Lemma 2.5. *If a regular system has a reducible monodromy group, then it is equivalent to a block upper-triangular regular system. The block structure is defined in accordance with the invariant subspaces and the diagonal blocks are of the minimal possible sizes, i.e. these blocks define systems with irreducible or one-dimensional monodromy groups.*

Lemma 2.5 is proved in Subsection 2.8.

Notation 2.6. Capital Latin letters denote $n \times n$-matrices or their blocks (the rare exceptions from this rule are specified); $I = \mathrm{diag}(1, \ldots, 1)$. The matrix having a single non-zero entry in position (i, j) which is equal to 1, is denoted by $E_{i,j}$. In all other cases double lowercase indices (subscripts) to matrices indicate their entries.

2.4 Some remarks on Fuchsian systems and the Riemann–Hilbert problem

The old version of the Riemann–Hilbert problem was quite natural – for given positions of the poles a Fuchsian system is completely defined by its matrices-residua $A_1, \ldots, A_p$ (one has $A_{p+1} = -A_1 - \cdots - A_p$) while its monodromy group is defined by the matrices $M_1, \ldots, M_p$ (see (4)), i.e. both objects depend locally on one and the same number pn^2 of parameters.

The negative answer to the problem in this version resembles the impossibility to cover a projective space by a single affine chart (although both objects have the same dimension). For fixed poles part of the monodromy groups not realized by such Fuchsian systems would be realized by Fuchsian systems with an additional apparent singularity at a given point the residuum at which is conjugate to $\mathrm{diag}(1, 0, \ldots, 0, -1)$. This will be another "affine chart". If there remain again monodromy groups which are not realizable, then one can try to realize them with one of the next "affine charts" – when the residuum of the apparent singularity is conjugate to $\mathrm{diag}(2, 0, \ldots, 0, -2)$, $\mathrm{diag}(1, 1, 0, \ldots, 0, -2)$, $\mathrm{diag}(1, 1, 0, \ldots, 0, -1, -1)$, $\mathrm{diag}(2, 0, \ldots, 0, -1, -1)$ etc. After finitely many such steps all monodromy groups will be realized.

The inconvenience of allowing additional apparent singularities is that one must specify not only the conjugacy class of the last matrix-residuum, but also the condition the monodromy operator around it to equal I. This is an algebraic condition on the matrices-residua. If one prefers allowing one regular non-Fuchsian singularity among the $p+1$ fixed poles of the system, then similar conditions arise when one requires the system to be regular at the last point. If numerical computations are to be performed, then in this case approximation errors will make with probability 1 the system irregular at the non-Fuchsian singularity, which will be a qualitative change in the behaviour of its solutions.

If the poles of a Fuchsian system are regarded as parameters of a deformation, then the condition for such a deformation to be isomonodromic (i.e. the monodromy group to remain the same) are a system of relatively simple differential equations, see [27], [23], [25] and [2]. In [25] and [2] the complete integrability of this system is proved as well as the *Painlevé property* of its solutions – to have only poles as movable singularities. Other properties of its solutions and properties of isomonodromic confluences of Fuchsian singularities are studied in [8], [9], [10], [11].

Fuchsian systems appear in the theory of holonomic quantum fields, see [27].

Closely related to the Riemann–Hilbert problem is the *Deligne–Simpson problem*: *Give necessary and sufficient conditions upon the* $(p+1)$*-tuples of conjugacy classes* $c_j \subset gl(n,\mathbb{C})$ *or* $C_j \subset \mathrm{GL}(n,\mathbb{C})$ *so that there exist irreducible* $(p+1)$*-tuples of matrices* $A_j \in c_j$, $A_1 + \cdots + A_{p+1} = 0$, *or* $M_j \in C_j$, $M_1 \ldots M_{p+1} = I$. Paper [20] is a survey on the Deligne–Simpson problem.

2.5 Levelt's result

In [24] A. H. M. Levelt describes the form of the solution to a regular system at its pole. We refer the reader to [5] as well.

Theorem 2.7. *In the neighbourhood of a pole the solution to a regular linear system* (1) *is representable in the form*

$$X = U_j(t-a_j)(t-a_j)^{D_j}(t-a_j)^{E_j}G_j \tag{5}$$

where U_j *is holomorphic in a neighbourhood of the pole* a_j, $D_j = \mathrm{diag}(\varphi_{1,j},\ldots,\varphi_{n,j})$, $\varphi_{n,j} \in \mathbb{Z}$, $G_j \in \mathrm{GL}(n,\mathbb{C})$. *The matrix* E_j *is in upper-triangular form and the real parts of its eigenvalues belong to* $[0,1)$ (*by definition,* $(t-a_j)^{E_j} = e^{E_j \ln(t-a_j)}$). *The numbers* $\varphi_{k,j}$ *satisfy the condition* (7) *formulated below. They are valuations in the eigenspaces of the monodromy operator* M_j (*i.e. in the maximal subspaces invariant for* M_j *on which it acts as an operator with a single eigenvalue*).

System (1) *is Fuchsian at* a_j *if and only if*

$$\det U_j(0) \neq 0. \tag{6}$$

We formulate the condition on $\varphi_{k,j}$. Let E_j have one and the same eigenvalue in the rows with indices $s_1 < s_2 < \cdots < s_q$. Then one has

$$\varphi_{s_1,j} \geq \varphi_{s_2,j} \geq \cdots \geq \varphi_{s_q,j} \tag{7}$$

Remarks 2.8. 1) Denote by $\beta_{k,j}$ the diagonal entries (i.e. the eigenvalues) of the matrix E_j. Then in the case of a Fuchsian system the sums $\beta_{k,j} + \varphi_{k,j}$ are the eigenvalues of the matrix-residuum A_j at a_j.

2) One has (up to conjugacy) $\exp(2\pi i E_j) = M_j$.

Proposition 2.9. *Let system* (1) *be regular, but not Fuchsian at a_j, i.e. in representation* (5) *one has* $\det U_j(0) = 0$. *Then there exists a transformation* $X \mapsto P(1/(t-a_j))X$, $\det P \equiv \text{const} \neq 0$, *$P$ is a matrix-polynomial of* $1/(t-a_j)$, *which brings the solution to system* (1) *to form* (5) *at a_j with* (6) *fulfilled;* (7) *might not be fulfilled* (*if not, then the integers $\varphi_{k,j}$ do not have the meaning of valuations, see Theorem* 2.7).

Proof. It is shown in [4] that a holomorphic matrix-function $U_j(t-a_j)$ can be represented as

$$P(1/(t-a_j))U^0(t-a_j)(t-a_j)^K$$

where $K = \text{diag}(k_1, \ldots, k_n)$, $k_\nu \in \mathbb{Z}$, $U^0 \in \mathcal{O}(t-a_j) \otimes \text{GL}(n, \mathbb{C})$ and P is a matrix-polynomial of $1/(t-a_j)$, $\det P \equiv \text{const} \neq 0$ (this statement is also known as Sauvage's lemma). The matrix P defines the necessary transformation, the matrix K shows how the integers $\varphi_{k,j}$ change. □

Remark 2.10. It is clear that the transformation $X \mapsto PX$ preserves form (5) of the solution to system (1) at the other poles (a_k, $k \neq j$) and conditions (6) and (7) if system (1) is Fuchsian at a_k for $k \neq j$.

Proposition 2.11. *A matrix-function of form* (5) *with condition* (6) *fulfilled, E_j in upper-triangular Jordan normal form and condition* (7) *not fulfilled is* (*locally, at a_j*) *a solution to a regular but not Fuchsian linear system at a_j.*

Let $\varphi = \max |\varphi_{k,j} - \varphi_{k+1,j}|$, *the maximum being taken over all $\varphi_{k,j}$ such that*

1) $\varphi_{k,j} < \varphi_{k+1,j}$,
2) *E_j has a Jordan block in the rows with indices $s, s+1, \ldots, s+q$, $s \leq k < k+1 \leq s+q$.*

Then system (1) *has a pole of order $\varphi + 1$ at a_j.*

Proof. Regularity follows from form (5). The proposition is checked directly when $U_j \equiv I$ and E_j consists of one Jordan block. For the general case the result follows easily from rule (3) and we prefer to let the reader complete the proof oneself. □

2.6 A more precise formulation of the basic result

If system (1) is Fuchsian at $a_1, \dots, a_{p+1}$ and if $\beta_{k,j}$ denote as in the previous subsection the diagonal entries of the matrix E_j from (5), then one has

$$\sum_{k=1}^{n}\sum_{j=1}^{p+1} (\varphi_{k,j} + \beta_{k,j}) = 0, \tag{8}$$

see [5]. Call for j fixed, $1 \le j \le p+1$, *admissible* any set of integers $\varphi_{k,j}$, $k = 1, \dots, n$ satisfying condition (7).

Theorem 2.12. *Suppose that the set of different points $a_1, \dots, a_{p+1} \in \mathbb{C}P^1$ is fixed, the subgroup of* $\mathrm{GL}(n, \mathbb{C})$ *generated by the matrices $M_1, \dots, M_p$ is irreducible, and for each j, $j = 1, \dots, p+1$ the set of integers $\{\tilde{\varphi}_{k,j}\}$, $k = 1, \dots, n$ is given and admissible* (*with respect to the matrices E_j defined in Theorem* 2.7). *Suppose also that for $j = 1$ one has*

$$\tilde{\varphi}_{s_k,1} \ge \tilde{\varphi}_{s_{k+1},1} + N, \;\; N = (n-1)(p+1)$$

whenever the numbers $\tilde{\varphi}_{k,1}$, $\tilde{\varphi}_{k+1,1}$ correspond to one and the same eigenvalue of E_1 (*i.e. if E_1 has one and the same eigenvalue in the rows with indices $s_1 < s_2 < \cdots < s_q$*), *and that equality* (8) *is fulfilled. Then there exists a Fuchsian system on $\mathbb{C}P^1$ with poles at $a_1, \dots, a_{p+1}$ and only there for which*

1) *the monodromy operators corresponding to $a_1, \dots, a_{p+1}$ are $M_1, \dots, M_{p+1}$ where $M_{p+1} = (M_1 \dots M_p)^{-1}$, see Subsection* 2.2;

2) *the integers $\varphi_{k,j}$ in* (5), *$k = 1, \dots, n$; $j = 2, \dots, p+1$, are equal to $\tilde{\varphi}_{k,j}$;*

3) *the integers $\varphi_{k,1}$ in* (5) *differ from the corresponding integers $\tilde{\varphi}_{k,1}$ by no more than N and conditions* (6), (7) *are fulfilled at $a_1, \dots, a_{p+1}$.*

Remarks 2.13. 1) It is possible to find sets of points $a_1, \dots, a_{p+1} \in \mathbb{C}P^1$ and regular systems with poles there (and only there) which are not equivalent to Fuchsian ones (with the same set of poles and with no other poles on $\mathbb{C}P^1$), see Theorem 3 in [5]. Theorem 2.12 shows that their monodromy groups are reducible. A. A. Bolibrukh has found (for $n = 4$) an example of a reducible subgroup of $\mathrm{GL}(4, \mathbb{C})$ which is not the monodromy group of a Fuchsian system on $\mathbb{C}P^1$ for any set of poles, see [7].

2) It seems possible to estimate the number N better, see 2.7 H). This, in turn, could provide a better estimation for the order of the pole at a_1 necessary to realize every finitely generated subgroup of $\mathrm{GL}(n, \mathbb{C})$ as a monodromy group of a regular system on $\mathbb{C}P^1$ which is Fuchsian at $a_2, \dots, a_{p+1}$; see Proposition 2.11 and [1].

2.7 Proof of Theorem 2.12

(A) *Plan of the proof.* The proof consists of four steps. It follows from the correct part of Plemelj's wrong proof of the Riemann–Hilbert problem in [26] that for any finitely generated subgroup of GL$(n, \mathbb{C})$ there exists a regular system on $\mathbb{C}P^1$ for which this subgroup is its monodromy group and which is Fuchsian at $a_2, \ldots, a_{p+1}$; we do not claim that it is Fuchsian at a_1, but by Proposition 2.9 we can assume that in form (5) of the solution at a_1 condition (6) is fulfilled (condition (7) might not hold). The first step (described in (B)) consists in changing the numbers $\varphi_{k,j}$, $k = 1, \ldots, n$; $j = 2, \ldots, p+1$ to the desired admissible set $\{\tilde{\varphi}_{k,j}\}$.

In the second step (see (C)) we perform a transformation (2) with an additional (apparent) singularity (namely – a pole) at $a^0 \neq a_1, \ldots, a_{p+1}$ (and with W holomorphic and holomorphically invertible for $t \neq a^0, a_1$). After this transformation one has

$$\varphi_{k,1} \geq \varphi_{k+1,1} + N, \quad N = (n-1)(p+1), \; k = 1, \ldots, n-1 \tag{9}$$

and conditions (6) and (7) are fulfilled at $a_1, \ldots, a_{p+1}$. Hence, system (1) has become Fuchsian at $a_1, \ldots, a_{p+1}$, but it has an apparent singularity at a^0. Recall that "apparent" means the monodromy at a^0 to be trivial (i.e. equal to I). System (1) is regular at a^0 because W has only a pole there.

The third step consists in performing a transformation (2) with det W = const $\neq 0$, W being holomorphic for $t \neq a^0$ and having a pole at a^0, see (D), (E), (F). This transformation makes the singularity at a^0 Fuchsian and after the transformation the solution to system (1) at a^0 can be represented in form (5), condition (6) being fulfilled, with $E^0 = 0$, $D^0 = \text{diag}\,(\varphi_1, \ldots, \varphi_n)$, where $\varphi_j - \varphi_{j+1} \leq p+1$. Form (5) (at $a_1, \ldots, a_{p+1}$) and the integers $\varphi_{k,j}$, $k = 1, \ldots, n$; $j = 1, \ldots, p+1$ do not change but only the matrices U_j, $j = 1, \ldots, p+1$ do; condition (6) is preserved at $a_1, \ldots, a_{p+1}$.

The fourth step consists in performing a transformation (2) with W holomorphic and holomorphically invertible for $t \neq a^0, a_1$, see (G). After this transformation the apparent singularity at a^0 disappears, the integers $\varphi_{k,j}$, $k = 1, \ldots, n$; $j = 2, \ldots, p+1$ are preserved (and form (5) with condition (6) at $a_2, \ldots, a_{p+1}$ as well). At a_1 one has form (5), condition (6) and the integers $\varphi_{k,1}$ are changed, but condition (7) holds and, hence, system (1) is Fuchsian on $\mathbb{C}P^1$.

Remark 2.14. It is possible to perform the third step, due to the irreducibility of the monodromy group of system (1). The system remains Fuchsian at a_1 after the fourth step due to condition (9).

(B) Let a system (1) be given, let $\{\varphi_{k,j}\}$ be its set of diagonal entries of the matrices D_j, $j = 1, \ldots, p+1$.

Fix j, $2 \leq j \leq p+1$. Let s be the minimal value of k for which one has $\varphi_{k,j} \neq \tilde{\varphi}_{k,j}$. Suppose that $U_j(0) = I$ (this can be achieved by a transformation (2) with a constant matrix W, det $W \neq 0$). Let $\varphi_{s,j} > \tilde{\varphi}_{s,j}$ (respectively, $\varphi_{s,j} < \tilde{\varphi}_{s,j}$).

Then we perform the transformation

$$X \mapsto \text{diag}\,(1, \ldots, 1, \chi, 1, \ldots, 1)X$$

with $\chi = (t - a_1)/(t - a_j)$ (respectively with $\chi = (t - a_j)/(t - a_1)$), χ stands in the s-th row. After this we represent the matrix $(t - a_j)^{D_j}$ in the form

$\mathrm{diag}(1, \ldots, 1, \tilde{\chi}, 1, \ldots, 1)(t-a_j)^{D'_j}$, where $\tilde{\chi} = t - a_j$ (respectively $\tilde{\chi} = (t-a_j)^{-1}$) stands in the s-th row. So, the matrix-function $\mathrm{diag}\,(1, \ldots, 1, \chi, 1, \ldots, 1)U_j(t-a_j)$ $\mathrm{diag}\,(1, \ldots, 1, \tilde{\chi}, 1, \ldots, 1)$ is holomorphic and holomorphically invertible at a_j, and $\varphi_{s,j}$ has decreased (has increased) by 1. After a finite number of such procedures we obtain the desired admissible set of integers $\{\varphi_{k,j}\} = \{\tilde{\varphi}_{k,j}\}$, $k = 1, \ldots, n$; $j = 2, \ldots, p+1$.

(C) Suppose that $U_1(0) = I$ (see Proposition 2.9).

Similarly to (B), we perform a finite number of procedures

$$X \mapsto \mathrm{diag}\,(1, \ldots, 1, \kappa, 1, \ldots, 1)X$$

with $\kappa = (t-a_1)/(t-a^0)$ or $\kappa = (t-a^0)/(t-a_1)$ where $a^0 \neq a_1, \ldots, a_{p+1}$ is fixed and the position of κ can vary so that to change the set $\{\varphi_{k,1}\}$ to the given one $\{\tilde{\varphi}_{k,1}\}$; each procedure is followed by a transformation (2) with $W = U_1^{-1}(0)$ (to have $U_1(0) = I$ after each procedure). As a result we obtain a system (1) which is Fuchsian at $a_1, \ldots, a_{p+1}$ and regular at a^0; the monodromy operator at a^0 is equal to I.

(D) After (C) the solution to system (1) at a^0 can be represented in form (5) with $E^0 = 0$. Using Proposition 2.9, we transform system (1) into one which is Fuchsian at $a^0, a_1, \ldots, a_{p+1}$; indeed, condition (7) is vacuous at a^0.

Lemma 2.15. *Consider under these assumptions the matrix-function* $U^0(t-a^0)$ (*taken from form* (5) *of the solution to system* (1) *at* a^0), $U^0(0) = I$. *Then it has at least one non-diagonal entry in each row and one non-diagonal entry in each column and one entry in each set* Ω *described below such that the order of the zero of this entry as a holomorphic function of* $(t-a^0)$ *for* $t = a^0$ *is not greater than* $(p-1)$.

Define the sets Ω. Represent the set of n natural numbers $\{1, \ldots, n\}$ in the form $\mathcal{J}_1 \cup \mathcal{J}_2$, $\mathcal{J}_1 \cap \mathcal{J}_2 = \emptyset$, $\mathcal{J}_1 \neq \emptyset$, $\mathcal{J}_2 \neq \emptyset$. For every such pair $(\mathcal{J}_1, \mathcal{J}_2)$ the corresponding set Ω consists of the entries of $U^0(t-a^0)$ in the rows with indices from $\mathcal{J}_1$ and in the columns with indices from $\mathcal{J}_2$.

Proof. 1^o. *Suppose that there exists a pair* $(\mathcal{J}_1, \mathcal{J}_2)$, *for which the corresponding set* Ω *does not contain the entry claimed by the lemma. Without loss of generality we can put* $\mathcal{J}_1 = \{1, \ldots, \nu\}$, $\mathcal{J}_2 = \{\nu+1, \ldots, n\}$, $1 \leq \nu \leq n-1$. *This implies that* Ω *is the right upper* $\nu \times (n-\nu)$*-corner of* $U^0(t-a^0)$.

Then the matrix-function $A(t)$, *see* (1), *has a zero at* a^0 *of order at least* p *in each entry of its upper* $\nu \times (n-\nu)$*-corner.*

Indeed, let the solution to system (1) be representable at a^0 in the form

$$U^0(t-a^0)(t-a^0)^{D^0}, \quad D^0 = \mathrm{diag}(\varphi_1, \ldots, \varphi_n), \quad U^0(0) = I.$$

The matrix-function $(t-a^0)^{D^0}$ is a solution to the equation

$$\dot{X} = (t-a^0)^{-1}\,\mathrm{diag}(\varphi_1, \ldots, \varphi_n)X$$

The matrix-function $U^0(t-a^0)(t-a^0)^{D^0}$ is a solution to an equation obtained from the last one by applying rule (3) with $W = U^0$. This proves the claim.

2^o. Every entry of the matrix of 1-forms $A(t)dt$, belonging to the right upper $\nu \times (n-\nu)$-corner, see (1), has a zero of order at least p at a^0 and a pole of order at most 1 at $a_1, \ldots, a_{p+1}$ and is holomorphic for $t \neq a^0, a_1, \ldots, a_{p+1}$. But the number of zeros of a meromorphic differential 1-form on $\mathbb{C}P^1$ is equal to the number of poles (counted with their multiplicities) minus two. Hence, the right upper $\nu \times (n-\nu)$-corner of $A(t)dt$ is identically equal to zero and the monodromy group of the system (1) must be reducible which is a contradiction. The lemma is proved (note that the non-diagonal entries of a row (or of a column) of $U^0(t-a^0)$ are a set Ω for $\mathcal{J}_1$ (or $\mathcal{J}_2$) consisting of one entry only). □

(E) We describe *Procedure* $\mathcal{P}(\Omega)$ here; it is similar to the procedures used by A. A. Bolibrukh in [5]. Consider the matrix U^0 (from form (5) of the solution to system (1) at a^0). Let $U^0(0) = I$ and let Ω be a fixed set defined as in (D); without loss of generality we assume that $\mathcal{J}_1 = \{1, \ldots, \nu\}$, $\mathcal{J}_2 = \{\nu+1, \ldots, n\}$, $1 \leq \nu \leq n-1$.

Let $U^0_{\alpha\beta} \in \Omega$ be an entry with the lowest possible order of the zero at a^0 (hence, lower or equal to $p-1$). Denote this order by q.

Lemma 2.16. *There exists a transformation* (2) *with* W *holomorphic and holomorphically invertible for* $t \neq a^0$, $\det W = \text{const} \neq 0$, W *having a pole at* a^0*, such that after this transformation the solution to system* (1) *is representable in form* (5) *at* $a^0, a_1, \ldots, a_{p+1}$ *with conditions* (6) *and* (7) *fulfilled. This transformation does not change the integers* $\varphi_{k,j}$, $k = 1, \ldots, n$; $j = 1, \ldots, p+1$. *It preserves the integers* φ_k, $k \neq \alpha, \beta$ *and one has* $\varphi_\alpha \mapsto \varphi_\alpha - q$, $\varphi_\beta \mapsto \varphi_\beta + q$.

The transformation described by the lemma is called Procedure $\mathcal{P}(\Omega)$.

Proof. 1^o. Decompose the matrix $U^0(t-a^0)$ as follows:

$$U^0 = \begin{pmatrix} M & \Omega \\ N & P \end{pmatrix}, \quad M \text{ is } \nu \times \nu.$$

2^o. Subtract the α-th row multiplied by suitably chosen polynomials of $1/(t-a^0)$ of degree $\leq q$ from the rows with indices in $\mathcal{J}_2$. We choose the polynomials so that after the subtraction all the entries of the β-th column should have at a^0 a zero of order $\geq q$. No poles will appear in the P-block of U^0 (due to the choice of $U^0_{\alpha\beta}$) but poles of order $\leq q$ will appear in the N-block; a pole of order exactly q will appear only in $U^0_{\beta\alpha}$ (this follows from $U^0(0) = I$).

3^o. Subtract the rows with indices from $\mathcal{J}_1 \backslash \{\alpha\}$ multiplied by monomials of $1/(t-a^0)$ of degree $\leq q-1$ from the rows with indices from $\mathcal{J}_2$. The monomials are chosen such that the order of the poles in the N-block which are in the columns with indices from $\mathcal{J}_1 \backslash \{\alpha\}$ decreases. Such a choice is possible because $M \mid_{t=a^0} = I$. Note that this operation does not introduce poles in the P-block. After it the order of the zero of the entries of the β-th column at a^0 is at least 1, but might be less than q (if the number of the row of the entry belongs to $\mathcal{J}_2$).

4^o. Repeat the operation described in 2^o. This time we have to choose the polynomials of $1/(t-a^0)$ to be of degree $\leq q-1$, see 3^o.

This might introduce poles of order $\leq q-2$ in the $\{N$-block $\}\backslash\{\ \alpha$-th column $\}$.

5^o. Repeat the operation described in 3^o. The monomials of $1/(t-a^0)$ have to be chosen of degree $\leq q-2$.

6^o. Repeating (as in 4^o and 5^o) the operations described in 2^o and 3^o (constantly decreasing the degree of the polynomials and monomials of $1/(t-a^0)$) we come to a matrix U^0 which has:

1) units on the diagonal except in $U^0_{\beta\beta}$

2) poles of order $\leq q$ in $\{N$-block $\}\cap\{\alpha$-th column $\}$ and only there

3) a pole of order exactly q only in $U^0_{\beta\alpha}$

4) zeros of order $\geq q$ (for $t=a^0$) in the β-th column and a zero of order exactly q in $U^0_{\alpha\beta}$.

7^o. Represent the matrix U^0 as

$$U^0 = U'^0 \text{ diag } (1,\ldots,1,(t-a^0)^{-q},1,\ldots,1,(t-a^0)^q,1,\ldots,1)$$

where $(t-a^0)^{-q}$ is in the α-th row, $(t-a^0)^q$ is in the β-th row; it is clear that U'^0 is holomorphic at a^0 and $\det U'^0\ |_{t=a^0}\neq 0$. Setting $D'^0 = D^0 +$ diag $(0,\ldots,0,-q,0,\ldots,0,q,0,\ldots,0)$, we obtain the proof of the lemma. □

(F) We describe an algorithm here based on Procedure $\mathcal{P}(\Omega)$. It changes the numbers φ_k in the following way: denote by $(i_1,\ldots,i_n)$ any permutation of the numbers $(1,\ldots,n)$ such that

$$\varphi_{i_1}\geq\varphi_{i_2}\geq\cdots\geq\varphi_{i_n}$$

We want to achieve the following condition:

$$\varphi_{i_k}\leq\varphi_{i_{k+1}}+p-1,\quad k=1,\ldots,n-1. \tag{10}$$

The algorithm consists of two steps:

Step 1. If (10) is fulfilled, then stop. If not, then go to Step 2.

Step 2. Choose the smallest k for which one has $\varphi_{i_k}>\varphi_{i_{k+1}}+p-1$. Perform Procedure $\mathcal{P}(\Omega)$ with $\mathcal{J}_1=\{i_1,\ldots,i_k\}$, $\mathcal{J}_2=\{i_{k+1},\ldots,i_n\}$. Go to Step 1.

Lemma 2.17. *The algorithm stops after a finite number of steps.*

Proof. 1^o. The number φ_{i_1} cannot increase and the number φ_{i_n} cannot decrease (this easily follows from $q\leq p-1$, q is as in Lemma 2.16). Condition (8) implies that

$$\sum_{j=1}^{p+1}\sum_{k=1}^{n}(\varphi_{k,j}+\beta_{k,j})+\sum_{k=1}^{n}\varphi_k=0.$$

One also has

$$\sum_{j=1}^{p+1} \sum_{k=1}^{n} (\varphi_{k,j} + \beta_{k,j}) = 0$$

from the conditions of the theorem. Hence, $\sum_{k=1}^{n} \varphi_k = 0$.

2^o. The last equality (and $\varphi_{i_1} \searrow, \varphi_{i_n} \nearrow$) implies that the numbers φ_k remain bounded. It follows from 1^o that after a finite number of steps, the numbers φ_{i_1} and φ_{i_n} cease to change. Hence, after a finite number of steps one must have $\varphi_{i_1} \leq \varphi_{i_2} + p - 1$, otherwise one must perform $\mathcal{P}(\Omega)$ with $\mathcal{J}_1 = \{i_1\}$ which decreases φ_{i_1} (this follows from $q \leq p - 1$, see Lemmas 2.15 and 2.16).

Having stabilized φ_{i_1}, one can stabilize φ_{i_2} after a finite number of steps, otherwise one has $\varphi_{i_1} > \varphi_{i_2} + p - 1$ and one must perform $\mathcal{P}(\Omega)$ with $\mathcal{J}_1 = \{i_1\}$ and this decreases φ_{i_1} etc. □

(G) After (F) one has $\sum_{k=1}^{n} \varphi_k = 0$. The equality $\sum_{k=1}^{n} \varphi_k = 0$ implies that the minimal φ_k is non-positive. Let it be equal to φ. Perform the transformation

$$X \mapsto [(t - a^0)/(t - a_1)]^{-\varphi} X. \tag{11}$$

This transformation increases φ_k by $|\varphi|$, $k = 1, \ldots, n$ and decreases $\varphi_{k,1}$ by $|\varphi|$, $k = 1, \ldots, n$. After it the minimal φ_k is equal to 0 and the biggest one is not greater than N.

We describe *Procedure* $\mathcal{Q}$ here.

Let $U^0(0) = I$ (this can be achieved by a transformation (2) with $W =$ const $\in GL(n, \mathbb{C})$). Set $\mathcal{J}_0 = \{k \in \mathbb{N} \mid \varphi_k > 0\}$. Perform the transformation

$$X \mapsto \text{diag}[((t - a_1)/(t - a^0))^{\delta_1}, \ldots, ((t - a_1)/(t - a^0))^{\delta_n}]X \tag{12}$$

where $\delta_k = 1$ for $k \in \mathcal{J}_0$ and 0 elsewhere. Change after that the matrix D^0 according to the rule

$$D^0 \mapsto D^0 - \text{diag}\,(\delta_1, \ldots, \delta_n).$$

One comes to a new pair (U^0, D^0) with U^0 holomorphic at a^0 and $\det U^0 \mid_{t=a^0} \neq 0$ by the assumption $U^0(0) = I$ above. In fact, one has

$$U^0 \mapsto \text{diag}[\triangle^{\delta_1}, \ldots, \triangle^{\delta_n}]U^0 \,\text{diag}((t - a^0)^{\delta_1}, \ldots, (t - a^0)^{\delta_n}), \quad \triangle = \frac{t - a_1}{t - a^0}.$$

Before transformation (12) let the matrix $U_1(t - a_1)$ have a non-degenerate $m \times m$-minor ($m = \#\mathcal{J}_0$) in the rows with indices from $\mathcal{J}_0$ and in the columns with indices from $\mathcal{J}_0'$ where $\mathcal{J}_0'$ contains m different integers from $[1, n]$; $\det U_1(0) \neq 0$, hence, such a minor exists. Then after transformation (12) the matrix-function U_1 becomes

$$U_1^* = \text{diag}[((t - a_1)/(t - a^0))^{\delta_1}, \ldots, ((t - a_1)/(t - a^0))^{\delta_n}]U_1$$

and it can be represented as $U_1^* = U_1'(t - a_1)\,(t - a_1)^D$, $D = \text{diag}(\varepsilon_1, \ldots, \varepsilon_n)$, $\varepsilon_k = 1$ for $k \in \mathcal{J}_0'$ and 0 elsewhere. Here U_1' is holomorphic in a punctured neighbourhood of a_1, $\det U_1'$ has a finite non-zero limit for $t \to a_1$, U_1' has poles of order ≤ 1 in $(U_1')_{\xi\eta}$, $\xi \in \{\{1,\ldots,n\}\backslash\mathcal{J}_0\}$, $\eta \in \mathcal{J}_0'$. Set $D_1 \mapsto D_1 + D$. Now in some more details.

Assume (without loss of generality) that $\mathcal{J}_0 = \mathcal{J}_0' = \{1,\ldots,m\}$ (the assumption $\mathcal{J}_0 = \mathcal{J}_0'$ is necessary only for the easier representations of the procedures that follow). Represent U_1 in the form $\left(\begin{smallmatrix} P & Q \\ R & S\end{smallmatrix}\right)$, where P is $m \times m$. Then $\det P \neq 0$ and

$$U_1^* = \begin{pmatrix} ((t-a_1)/(t-a^0))P & ((t-a_1)/(t-a^0))Q \\ R & S \end{pmatrix},$$

$$U_1' = \begin{pmatrix} P/(t-a^0) & ((t-a_1)/(t-a^0))Q \\ R/(t-a^1) & S \end{pmatrix}.$$

Transformation (2) with

$$W = \begin{pmatrix} I & 0 \\ ((t-a^0)/(t-a_1))(-RP^{-1}) & I \end{pmatrix}$$

makes system (1) Fuchsian at $a^0, a_1, \ldots, a_{p+1}$, again, i.e. , its solution at these points has form (5) and conditions (6) and (7) are fulfilled (indeed, $\det W \equiv 1$). Perform a transformation (2) with $W = \text{const} \in \mathrm{GL}(n,\mathbb{C})$ to have again $U^0 \mid_{t=a^0} = I$. This completes Procedure $\mathcal{Q}$.

Obviously, after no more than $|\varphi| + \varphi_{i_1}$ times repeating Procedure $\mathcal{Q}$ (φ_{i_1} is defined after (F) and before transformation (B); one has $|\varphi| + \varphi_{i_1} \leq N$) one has $\varphi_k = 0$, $k = 1,\ldots,n$, i.e. the apparent singularity at a^0 disappears. After transformation (11) the numbers $\varphi_{k,1}$, $k = 1,\ldots,n$ decrease simultaneously by $|\varphi|$. After $|\varphi| + \varphi_{i_1}$ times performing Procedure $\mathcal{Q}$ each of them increases by no more than $|\varphi| + \varphi_{i_1}$, i.e. by no more than N (because $\varphi_{i_k} \leq \varphi_{i_{k+1}} + p - 1$). Before performing transformation (11) one has $\varphi_{k,1} \geq \varphi_{k+1,1} + N$, $k = 1,\ldots,n-1$. Hence, after transformation (11) these inequalities hold again and after $|\varphi| + \varphi_{i_1}$ times performing Procedure $\mathcal{Q}$ the inequalities $\varphi_{k,1} \geq \varphi_{k+1,1}$, $k = 1,\ldots,n-1$ hold. Hence, conditions (6) and (7) are fulfilled at $a_1,\ldots,a_{p+1}$. The theorem is proved. □

(H) To find a better estimation for the numbers $H(n,p)$ and N from Theorem 2.12 (such an estimation for $H(n,p)$ can be found in [1], p. 127) one can use the following lemma:

Lemma 2.18. *Suppose that one has $U^0(0) = I$ in form* (5) *of the solution to system* (1) *at a^0, see parts* (C) *and* (D). *Then none of the non-diagonal entries of the matrix $U^0(t-a^0)$ is identically equal to zero.*

Proof. Indeed, the opposite would mean that a component of a vector-column solution to system (1) is identically equal to zero. But then the monodromy group of system (1) is reducible (see Lemma 3.3 from [5]).

Lemma 2.18 gives further opportunities to change the integers φ_k and to decrease the difference $|\varphi_{i_1} - \varphi_{i_n}|$. □

2.8 Proof of Lemma 2.5

1^o. Let the monodromy group be a semi-direct sum, i.e. the monodromy operators of the system have the form $\left(\begin{smallmatrix} P & Q \\ 0 & R \end{smallmatrix}\right)$ where P is $k \times k$ and R is $(n-k) \times (n-k)$, $1 < k < n$; the case when the monodromy operators have a more complicated block upper-triangular structure can be treated in the same way, by induction with respect to the number of diagonal blocks. Represent the matrices U_j (see Theorem 2.7) in the form $U_j = \left(\begin{smallmatrix} P_j & Q_j \\ S_j & R_j \end{smallmatrix}\right)$, with the same sizes of the blocks.

2^o. Assume that for $(t - a_j)$ sufficiently small one has $\det P_j \neq 0$, $\det R_j \neq 0$, $j = 1, \ldots, p+1$ (this can be achieved by a linear equivalence (2) with $W = \text{const}$). Then in a neighbourhood of a_j the transformation

$$X \mapsto \begin{pmatrix} I & 0 \\ -S_j P_j^{-1} & I \end{pmatrix} X$$

makes the solution X of the system block upper-triangular.

3^o. The matrices U_j can be analytically continued along any path on the universal covering $\tilde{\Sigma}$ of $\mathbb{C}P^1 \backslash \{a_1, \ldots, a_{p+1}\}$. Hence, the matrices $(-S_j P_j^{-1})$ can be meromorphically continued on $\tilde{\Sigma}$ (det P_j can have isolated zeros). The continuations of $(-S_j P_j^{-1})$ for two different values of j must coincide at any point different from a_j, $j = 1, \ldots, p+1$, where both matrices are holomorphic. Indeed, let $\gamma^* X$ be the analytic continuation of X along a loop γ. Then

$$\gamma^* X = X \begin{pmatrix} P & Q \\ 0 & R \end{pmatrix}.$$

By formula (5) this can be represented as follows:

$$\gamma^* U_j(t-a_j)(t-a_j)^{D_j}(t-a_j)^{E_j} H = U_j(t-a_j)(t-a_j)^{D_j}(t-a_j)^{E_j} \begin{pmatrix} P & Q \\ 0 & R \end{pmatrix}$$

where the matrix H is block upper-triangular like $\left(\begin{smallmatrix} P & Q \\ 0 & R \end{smallmatrix}\right)$. Writing $\gamma^* U_j = \begin{pmatrix} P_j^* & Q_j^* \\ S_j^* & R_j^* \end{pmatrix}$, one has $S_j^*(P_j^*)^{-1} = S_j P_j^{-1}$.

Hence, there exists a transformation

$$X \mapsto TX, \quad T = \begin{pmatrix} I & 0 \\ V & I \end{pmatrix}$$

where V is meromorphic on $\mathbb{C}P^1$ and holomorphic at $a_1, \ldots, a_{p+1}$ which makes the system block upper-triangular (this transformation cannot be called "equivalence" because it can have poles different from a_j).

4^o. Let V have a pole at $c \neq a_j$, $j = 1, \ldots, p+1$. Represent $X' = TX$ in the form $\left(\begin{smallmatrix} P' & Q' \\ 0 & R' \end{smallmatrix}\right)$. Similarly to the fourth step from the proof of Theorem 2.12 we delete the apparent singularities from all the points $c \neq a_1, \ldots, a_{p+1}$ where the system has poles. This can be done by a block upper-triangular transformation which is a finite sequence of transformations of the kind

$$X \mapsto \operatorname{diag}(\kappa_1, \ldots, \kappa_n)X, \quad \kappa_j = 1 \text{ or } \kappa_j = ((t-c)/(t-a_1))^{\pm 1}$$

or of the kinds

$$X \mapsto \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} X \quad \text{or} \quad X \mapsto \begin{pmatrix} I & R \\ 0 & I \end{pmatrix} X$$

where P, Q, R can have poles at the points a_j or at the apparent singularity c and $\det P \equiv a \neq 0$, $\det Q \equiv b \neq 0$. We leave the details to the reader.

2.9 Proof of Theorem 2.4

1^o. The proof uses the same ideas as the ones of the proof of Theorem 2.12. Suppose that a reducible regular system is transformed to a block upper-triangular form (see Theorem 2.5). As in the proof of Theorem 2.12 we first bring the system to one with $\varphi_{k,j} = 0$ for $j = 1, \ldots, p$, see Subsection 2.7 part (B); here and further all transformations can be chosen preserving the block upper-triangular form.

2^o. After this we create an apparent singularity at some point $a^0 \neq a_1, \ldots, a_{p+1}$ in order to achieve the condition

$$\varphi_{k,p+1} - \varphi_{k+1,p+1} = N', \quad N' = (m-1)(p+1)$$

where m is the maximal size of a diagonal block, see Subsection 2.7 part (C). The last equalities hold only for $\varphi_{k,p+1}$, $\varphi_{k+1,p+1}$ corresponding to one and the same diagonal block, for all diagonal blocks.

3^o. We kill the apparent singularity as in Subsection 2.7 parts (E), (F), (G). After this, for every invariant subspace corresponding to a diagonal block of the system

$$\sum_{k=s}^{s+r} \sum_{j=1}^{p+1} (\varphi_{k,j} + \beta_{k,j}) = 0,$$

(the block is in the rows with indices $s, \ldots, s+r$; indeed, the block itself is a Fuchsian system). One has

$$0 \leq \varphi_{k,p+1} - \varphi_{k+1,p+1} \leq 2N', \quad k = s, \ldots, s+r-1.$$

4^o. Consider two different diagonal blocks of the system and their numbers $\varphi_{k,p+1}$. It follows from Proposition 2.11 that the order of the pole at a_{p+1} of the system is not greater than $\max|\varphi_{k_1,p+1} - \varphi_{k_2,p+1}|$ where the row with index k_1 (resp. k_2) belongs to the first (resp. to the second) block and $\varphi_{k_1,p+1} < \varphi_{k_2,p+1}$.

5^o. It follows from 3^o that if $\varphi_{k,j} = 0$ for $j = 1, \ldots, p$ and if $0 \leq \mathrm{Re}\beta_{k,j} < 1$, then $\varphi_{k,p+1}$ cannot be all positive. Hence,

$$\varphi_{s,p+1} \leq (m+2)N' + \varphi_{s+r,p+1} \leq (m+2)N'$$

(because $\varphi_{s+r,p+1} < 0$). On the other hand, one has

$$\sum_{k=s}^{s+r} \varphi_{k,p+1} = -\sum_k \sum_j \beta_{k,j} = -\sum_{k=s}^{s+r} \sum_{j=1}^{p+1} \mathrm{Re}\beta_{k,j} > -(p+1)(r+1).$$

6^o. It follows from 3^o and 5^o that $\varphi_{k,p+1} \leq \varphi_{s+r,p+1} + 2(s+r-k)N'$; hence,

$$\begin{aligned}
\sum_{k=s}^{s+r} \varphi_{k,p+1} &\leq (r+1)\varphi_{s+r,p+1} + r(r+1)N', \\
\varphi_{s+r,p+1} &\geq \Big(\sum_{k=s}^{s+r} \varphi_{k,p+1} - r(r+1)N'\Big)/(r+1) \\
&> -(p+1) - rN' \geq -(p+1) - mN',
\end{aligned}$$

$$\begin{aligned}
\max|\varphi_{k_1,p+1} - \varphi_{k_2,p+1}| &\leq (m+2)N' + p + 1 + mN' \\
&= 2(m+1)(m-1)(p+1) + p + 1 \\
&= (2m^2 - 1)(p+1).
\end{aligned}$$

This number is maximal for $m = n - 1$ which proves the theorem. □

3 On invariants of matrix groups

3.1 Definitions

We write $\mathcal{M} = \{M_1, \ldots, M_p\}$ or $\mathcal{A} = \{A_1, \ldots, A_p\}$ to denote the matrix group (respectively, the matrix algebra) generated by the matrices $M_1, \ldots, M_p$ (respectively, $A_1, \ldots, A_p$). The results are formulated for groups only but they are valid (and proved in the same way) for algebras as well. We try to give the definitions in terms of matrices and their interpretations in terms of representations (recall that initially M_j were the matrices of the monodromy operators of a regular system which define an antirepresentation of $\pi_1(\mathbb{C}P^1 \setminus \{a_1, \ldots, a_{p+1}\})$ in $\mathrm{GL}(n, \mathbb{C})$ and A_j were its residua in the case when it is Fuchsian; we write "antirepresentation" because the monodromy operator corresponding to the concatenation of contours $\gamma_i \gamma_j$ equals $M_j M_i$).

Definition 3.1. Let the group $\{M_1, \ldots, M_p\} \subset \mathrm{GL}(n, \mathbb{C})$ be conjugate to one in block-diagonal form, the diagonal blocks (called *big blocks*) being themselves block upper-triangular; their block structure is defined by the sizes of their diagonal blocks (called *small blocks*). The restriction of the group to each of the small blocks is assumed to be an irreducible or one-dimensional matrix group of the corresponding size. The sizes of the big and small blocks are correctly defined modulo permutation of the big blocks (if we require that the sizes of the big blocks be the minimal possible). The big blocks are subrepresentations whose direct sum is the given representation (if there is more than one big block, then by definition the representation is called *completely reducible*). For each such representation its small blocks are its semisimple part in its Levi decomposition and its blocks above the diagonal are its radical (we say also its *nilpotent part*).

Definition 3.2. Call a *special pair* a pair of equivalent subrepresentations, i.e. a pair of small blocks of equal size l such that if M_j^1, M_j^2 are the restrictions of the matrices M_j to them, then there exists a matrix $Q \in \mathrm{GL}(l, \mathbb{C})$ such that $Q^{-1}M_j^1 Q = M_j^2$, $j = 1, \dots, p$. The *reducibility pattern* of the group $\{M_1, \dots, M_p\}$ is defined by the number and sizes of the small and big blocks and the position of the small blocks and their special pairs.

Example 3.3. Let $\mathcal{M}$ be blocked as follows:

$$\begin{pmatrix} A & C & 0 \\ 0 & B & 0 \\ 0 & 0 & D \end{pmatrix}$$

where the restrictions of $\mathcal{M}$ to the blocks A, B and D are irreducible matrix groups and it is impossible by conjugating the group $\mathcal{M}$ to obtain the condition $C = 0$. Then the reducibility pattern of $\mathcal{M}$ has two big and three small blocks (namely, $\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$, D and A, B, D). If in addition one has $M_j|_B = Q^{-1}M_j|_D Q$, then the pair (B, D) is special.

The following two lemmas will be used in finding the normal form of the centralizer in the generic case.

Lemma 3.4. *The equation*

$$\sum_{j=1}^{p}(P_j Z_j - Z_j R_j) = L$$

where the p-tuples of matrices $P_1, \dots, P_p \in gl(k, \mathbb{C})$ and $R_1, \dots, R_p \in gl(l, \mathbb{C})$ are irreducible, the unknown matrices Z_j being $k \times l$, has a solution for every $k \times l$-matrix L if and only if the pair (P, R) is not special (i.e. one does not have simultaneously $k = l$ and $P_1 = Q^{-1}R_1 Q, \dots, P_p = Q^{-1}R_p Q$ for some matrix $Q \in \mathrm{GL}(k, \mathbb{C})$).

The lemma is proved in [17]. Obviously, if the pair (P, R) is special and if $P_j = R_j$, $j = 1, \dots, p$, then for the solvability of the equation one must have $\mathrm{tr}L = 0$. This is sufficient as well, see [17]. From the proof of Lemma 3.4 in [17] one can deduce the "dual" statement:

Lemma 3.5. *Let the matrices P_j, R_j be defined as in Lemma* 3.4. *Then the conditions*

$$P_j Z - Z R_j = 0, \quad j = 1, \dots, p$$

imply that $Z = 0$ if (P, R) is not a special pair and that $Z = \alpha I$, $\alpha \in \mathbb{C}$ if it is.

3.2 Normal forms in the generic case

In [18] normal forms under conjugation of a matrix algebra and its centralizer are found. In this normal form the algebra (as a linear space) is a direct sum of its semisimple part (i.e. its restriction to the set of small blocks of its reducibility pattern) and its nilpotent part (i.e. its restriction to the set of its blocks above the diagonal); the blocks of each matrix from its centralizer are scalar or zero. The restriction of the algebra to the set of blocks above the diagonal is described by relations of the type that certain linear combinations of such blocks have to be 0. (If the blocks in positions (i_1, j_1),..., (i_k, j_k) participate in such a combination, then the diagonal blocks P_{i_1},...,P_{i_k} are equal and so are the blocks P_{j_1},...,P_{j_k}.)

In the present text we consider normal forms under conjugation of the centralizer of such an algebra in the generic case defined below.

Definition 3.6. Suppose that the group $\mathcal{M} = \{M_1, \dots, M_p\}$ is in block upper-triangular form with one big block only and suppose that any matrix $S \in \mathrm{GL}(n, \mathbb{C})$ such that the group $S^{-1}\mathcal{M}S$ is again block upper-triangular is itself block upper-triangular; by this condition we define the *generic case*. A *superdiagonal* of a matrix L is the set of its entries $L_{i,j}$, the difference $i - j$ being constant and non-positive. In the same way one defines a *superdiagonal of blocks* for a block upper-triangular matrix. Call first superdiagonal the set of diagonal blocks and last superdiagonal the block in the right upper corner.

Theorem 3.7. *In the generic case the centralizer $\mathcal{Z}(\mathcal{M})$ of the group $\mathcal{M}$ contains non-scalar matrices if and only if the reducibility pattern of the group $\mathcal{M}$ is of the following form:*

$$\mathcal{M} = \begin{pmatrix} M^{\mathrm{i}} & M^{\mathrm{ii}} & M^{\mathrm{iv}} \\ 0 & M^{\mathrm{iii}} & M^{\mathrm{v}} \\ 0 & 0 & M^{\mathrm{vi}} \end{pmatrix}$$

with block upper-triangular matrices M^{i}, M^{iii}, M^{vi}, the matrices M^{ii}, M^{iii}, M^{v} might be absent, and either

A) $M^{\mathrm{i}} = M^{\mathrm{vi}} = M^0$

or

B)

$$\begin{pmatrix} M^{\mathrm{i}} & M^{\mathrm{ii}} \\ 0 & M^{\mathrm{iii}} \end{pmatrix} = \begin{pmatrix} M^{\mathrm{iii}} & M^{\mathrm{v}} \\ 0 & M^{\mathrm{vi}} \end{pmatrix} = M^0.$$

The cases A) *and* B) *do not differ in principle. We divide them only to show that there might be and there might be no intersection of the blocks M^0. One has*

$$\mathcal{Z}(\mathcal{M}) = \left\{\beta I + \sum \alpha_j D_j, \alpha_j \in \mathbb{C}, \beta \in \mathbb{C}\right\}$$

where $\{D_j\}$ is a subset of the set of matrices containing blocks equal to I on one superdiagonal of blocks and zeros elsewhere. Hence, this is a superdiagonal of square blocks and the matrices D_j contain units on one superdiagonal and zeros elsewhere.

Moreover, these matrices contain zeros outside $M^1 = \begin{pmatrix} M^{\text{ii}} & M^{\text{iv}} \\ M^{\text{iii}} & M^{\text{v}} \end{pmatrix}$. *If* D^1_j *denote their restrictions to the block* M^{iv} *in case* A) *or to the block* M^1 *in case* B), *then the centralizer* $\mathcal{Z}(M^0)$ *equals* $\left\{ \sum \alpha_j D^1_j, \alpha_j \in \mathbb{C} \right\}$. *A matrix from* $\mathcal{Z}(\mathcal{M})$ *can contain non-zero blocks only on the last k superdiagonals and on the first one where k is the number of rows and columns of blocks of* M^0.

Corollary 3.8. 1) *For any fixed number of small blocks in the generic case there exist finitely many normal forms for the centralizers of the matrix groups satisfying the conditions of the theorem.*

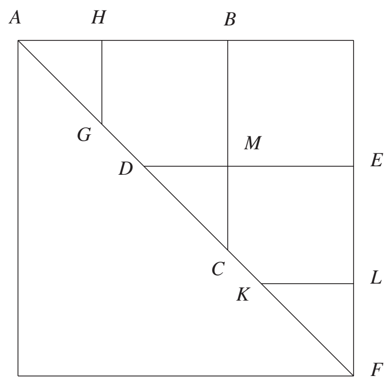
2) *If* $\mathcal{Z}(\mathcal{M})$ *is in its normal form, then the only relations between the blocks of* $\mathcal{M}$ *defined by the commutation relations are equalities between some blocks on one and the same superdiagonal.*

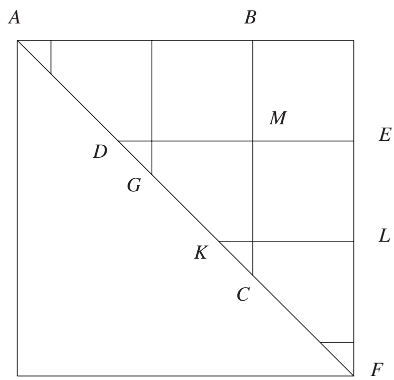
3) *All normal forms can be obtained recursively* (*by induction on the number of small blocks; for* M^0 *this number is smaller than the one corresponding to* $\mathcal{M}$).

4) *In the generic case the classification of the monodromy groups by the normal forms of their centralizers involves no modules.* (*In the next subsection we show that this is no longer true in the general case.*)

The corollary is evident.

Example 3.9. Let the block structure of $\mathcal{M}$ be represented schematically by the triangles in the figure below. Let $\triangle ABC = \triangle DEF = M_0$. Then $\triangle AHG = \triangle DMC = \triangle KLF$; $\triangle AHG$ and $\triangle KLF$ are obtained from $\triangle DMC = \triangle ABC \cap \triangle DEF$ by translation along the diagonal. A more complicated structure can be obtained if $\triangle DMC \cap \triangle KLF \neq 0$ etc., see the right part of the figure on which the four small triangles above the diagonal are equal.





Remark 3.10. The theorem is not correct if one requires only $\mathcal{Z}(\mathcal{M})$ to be block upper-triangular. The following example shows a situation where $\mathcal{Z}(\mathcal{M})$ is block

upper-triangular and there exist matrices S (which are not block upper-triangular) such that $S^{-1}\mathcal{M}S$ is again block upper-triangular.

Example 3.11. If

$$\mathcal{M} = \begin{pmatrix} P & 0 & R \\ 0 & P & T \\ 0 & 0 & P \end{pmatrix}$$

then

$$\mathcal{Z}(\mathcal{M}) = \left\{ \begin{pmatrix} \alpha I & 0 & \gamma I \\ 0 & \alpha I & \beta I \\ 0 & 0 & \alpha I \end{pmatrix},\ \alpha,\ \beta,\ \gamma \in \mathbb{C} \right\}.$$

The conjugation with

$$S = \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & I \end{pmatrix}$$

(not block upper-triangular) preserves the form of $\mathcal{M}$.

Proof of Theorem 3.7. 1^o. It follows from the conditions of the theorem that $\mathcal{Z}(\mathcal{M})$ consists only of block upper-triangular matrices. Let the reducibility pattern of $\mathcal{M}$ contain n' small blocks. We say that $\mathcal{M}$ is a group of type (n', k) if it can be conjugated to the form indicated in the theorem, with the same meaning of k (i.e. the number of small blocks of M^0).

2^o. Denote by $\mathcal{M}^u$ (resp. $\mathcal{C}^u$) and $\mathcal{M}^l$ (resp. $\mathcal{C}^l$) the restrictions of the matrices of $\mathcal{M}$ (resp. of $\mathcal{Z}(\mathcal{M})$) to the $(n'-1)$ upper rows and $(n'-1)$ left columns of blocks (respectively, to the $(n'-1)$ lower rows and $(n'-1)$ right columns of blocks). It is clear that the representations defined by $\mathcal{M}^u$ and $\mathcal{M}^l$ belong also to the generic case (but with $n'-1$ instead of n' small blocks).

Lemma 3.12. *If $\mathcal{M}^u$ (resp. $\mathcal{M}^l$) is of type $(n'-1, k_1)$ (resp. of type $(n'-1, k_2)$), then any matrix from $\mathcal{Z}(\mathcal{M})$ can contain non-zero entries only on the diagonal and on the last* $\min(k_1, k_2) + 1$ *superdiagonals of blocks.*

Proof. A^o. Note first that one has $\mathcal{C}^u \subset \mathcal{Z}(\mathcal{M}^u)$, $\mathcal{C}^l \subset \mathcal{Z}(\mathcal{M}^l)$. Suppose first (in A^0 – C^0) that $1 < k_2 < k_1$ (the case $1 < k_1 < k_2$ is treated similarly).

Suppose that $\mathcal{M}^u$ and $\mathcal{Z}(\mathcal{M}^u)$ are in the normal form indicated in the theorem. Then any matrix $C \in \mathcal{Z}(\mathcal{M}^u)$ is of the form $C = \left(\sum_{i=2}^{k_1+1} \alpha_i D_i^*\right) + \beta I^*, \alpha_i \in \mathbb{C}, \beta \in \mathbb{C}, D_i^*$ being matrices which contain zero blocks outside the $(n'-i+1)$-st superdiagonal of blocks and blocks equal to I on the $(n'-i+1)$-st one (excluding the blocks (i, n') which do not belong to $\mathcal{M}^u$); $I^* = I|_{\mathcal{M}^u}$.

B^o. Denote by $C' \in \mathcal{Z}(\mathcal{M})$ a matrix such that $C'|_{\mathcal{M}^u} = C$. If i_0 is the greatest index for which $\alpha_i \neq 0$, then $i_0 \leq k_1 + 1$ and for any block upper-triangular matrix $S \in \mathrm{GL}(n, \mathbb{C})$ the matrix $S^{-1}C'S$ has non-zero blocks only on the $(n'-i_0+1)$-st superdiagonal and/or above it.

C^o. If $k_2 + 1 < i_0 \leq k_1 + 1$, then the restriction to $\mathcal{M}^l$ of such a matrix C' (or the one of $S^{-1}C'S$) cannot belong to $\mathcal{Z}(\mathcal{M}^l)$ because it should not contain blocks on the $(n' - i_0 + 1)$-st superdiagonal. Hence, if $C \in \mathcal{C}^u$, then $i_0 \leq k_2 + 1$, i.e. C' and $S^{-1}C'S$ can contain non-zero entries only on the diagonal and on the last $k_2 + 1$ superdiagonals.

D^o. If $k_1 = k_2$, then the claim that only the last $k_1 + 1$ superdiagonals of $\mathcal{Z}(\mathcal{M}) \cap sl(n, \mathbb{C})$ can be non-zero is evident.

E^o. Set

$$M_j = \begin{pmatrix} A_j & C_j & \dots & N_j \\ 0 & B_j & \dots & Q_j \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & P_j \end{pmatrix}, \quad C = \begin{pmatrix} \alpha I & 0 & \dots & U \\ 0 & \alpha I & \dots & V \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha I \end{pmatrix}$$

(all other non-diagonal blocks except U and V of $C \in \mathcal{Z}(\mathcal{M})$ are presumed to be 0). Set $\alpha = 0$. If $k_1 = k_2 = 0$, then $V = 0$ and (A_j, P_j) is either a special pair (i.e. $A_j = P_j$, $j = 1, \dots, p$ and $U = \gamma I$) or it is not and $\mathcal{Z}(\mathcal{M}) = \{\beta I\}$. Hence, for $k_1 = k_2 = 0$ the lemma and the theorem are true.

F^o. Let $k_1 = 0$, $k_2 = 1$. The condition $k_2 = 1$ implies that the pair (B_j, P_j) is special. Then $B_jV - VP_j = 0$, $j = 1, \dots, p$. Recall Lemmas 3.4 and 3.5. Hence, either $V = 0$ (in this case $A_jU = UP_j$ and either the pair (A_j, P_j) is special, i.e. $A_j = P_j$, $U = \gamma I$, and the lemma and the theorem are true, or it is not special and $U = 0$, hence, the lemma and the theorem are true again) or $V = \beta I$, $\beta \in \mathbb{C}$.

Let $\beta \neq 0$. Set $\beta = 1$. Then $A_jU + C_jV = UP_j$, i.e. $C_j = UP_j - A_jU$. Hence, the conjugation of $\mathcal{M}$ with

$$\begin{pmatrix} I & U & \dots & 0 \\ 0 & I & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & I \end{pmatrix}$$

(all other non-diagonal blocks are 0) yields $C_j \mapsto 0$. This implies that the conjugation with the matrix permuting the first two rows and columns of blocks (which is not block upper-triangular) leaves $\mathcal{M}$ block upper-triangular. This is a contradiction with the condition of the theorem. Hence, $\beta = 0$.

The case $k_1 = 1$, $k_2 = 0$ is treated by analogy. The lemma is proved. □

3^o. We assume that $k_1 \geq k_2$ and that $\mathcal{M}^u$ and $\mathcal{Z}(\mathcal{M}^u)$ are in the form claimed by the theorem (the assumption is based on the fact that $\mathcal{M}^u$ satisfies the conditions of the theorem and has one small block less than $\mathcal{M}$; for reducibility patterns with one or two small blocks there is nothing to prove). Represent $\mathcal{M}$ and $C \in \mathcal{Z}(\mathcal{M})$ in the form

$$\mathcal{M} = \begin{pmatrix} A & \dots & B & U & & & & & \\ \vdots & \ddots & \vdots & \vdots & & & * & & \\ 0 & \dots & R & Y & & & & & \\ 0 & \dots & 0 & Z & & & & & \\ & & & & \ddots & & & & \\ & & & & & A & \dots & B & U' \\ & 0 & & & & \vdots & \ddots & \vdots & \vdots \\ & & & & & 0 & \dots & R & Y' \\ & & & & & 0 & \dots & 0 & Z' \end{pmatrix},$$

$$C = \begin{pmatrix} \beta I & & 0 & & & \zeta I & \dots & \eta I & \Phi \\ & \ddots & & & & \vdots & \ddots & \vdots & \vdots \\ & & \beta I & & & 0 & \dots & \zeta I & \Gamma \\ & & & \beta I & \dots & 0 & \dots & 0 & \Omega \\ & & & & \ddots & \vdots & & & \\ & & & & & \beta I & & 0 & \\ & 0 & & & & & \ddots & & \\ & & & & & & & \beta I & \\ & & & & & & & & \beta I \end{pmatrix}.$$

Formally, this form is valid for case A) of the theorem. For case B) the reasoning is analogous.

The condition $[M_j, C] = 0$ yields $Z_j\Omega - \Omega Z_j' = 0$, $j = 1, \dots, p$. Hence, by Lemma 3.5, either (Z_j, Z_j') is a special pair (i.e. $Z_j = Z_j'$ and $\Omega = \omega I$) or $\Omega = 0$. In the second case we have $\zeta = 0$ for any matrix $C \in \mathcal{Z}(\mathcal{M})$ (just by repeating the reasoning from the proof of the lemma), and we restart the consideration of $\mathcal{M}$ and C, assuming that $\mathcal{M}^u$ is of type (n', k') with $k' \leq k_1 - 1$ and that $\mathcal{M}^l$ is of type (n', k'') with $k'' \leq \min((k_1 - 1), k_2)$. (Such a restarting can occur only a finite number of times.) In other words, we require from $\mathcal{M}^u$ something less, i.e. we forget the condition $D_i^* \in \mathcal{Z}(\mathcal{M})$ for the greatest value of i.

4^o. Assume that $\zeta \neq 0$, $\omega \neq 0$. Conjugate $\mathcal{M}$ and $\mathcal{Z}(\mathcal{M})$ with the matrix $\mathrm{diag}(I, \dots, I, (\zeta/\omega)I)$. This yields $\omega = \zeta$.

There exists a conjugation of $\mathcal{M}$ and of C with a block upper-triangular matrix after which C will contain equal scalar blocks on the $(n' - k_2 + 1)$-st superdiagonal (the blocks ζI) and zeros elsewhere (the reader will easily prove the existence of this conjugation oneself). After this $C \in \mathcal{Z}(\mathcal{M})$ implies $M' = M^{\mathrm{vi}} = M^0$ (or $\begin{pmatrix} M^{\mathrm{i}} & M^{\mathrm{ii}} \\ 0 & M^{\mathrm{iii}} \end{pmatrix} = \begin{pmatrix} M^{\mathrm{iii}} & M^{\mathrm{v}} \\ 0 & M^{\mathrm{vi}} \end{pmatrix} = M^0$). Hence, the right upper corner of C (of the size of M^0) commutes with M_j^0, $j = 1, \dots, p$ and one reduces the problem to finding the normal form of the centralizer of M^0 whose size is smaller than $n \times n$. This implies all the claims of the theorem. □

3.3 The presence of moduli in the general case in the classification of the monodromy groups by the normal forms of their centralizers

Consider the subalgebra $\mathcal{A} \subset sl(n, \mathbb{C})$ of matrices commuting with

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad a \neq 0.$$

A direct computation shows that $\mathcal{A} = \mathbb{C}K \oplus \mathbb{C}L \oplus \mathbb{C}M$ with

$$K = \begin{pmatrix} 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The centralizer of $\mathcal{A}$ is spanned as a linear space by P, R and $M = PR$.

Proposition 3.13. *For two nearby values of the parameter a the corresponding algebras $\mathcal{A}$ are non-equivalent, i.e. are not obtained from one another by conjugation and linear change of the generators. Thus the parameter a is a module when these algebras are classified by the normal forms of their centralizers.*

Four is the least size of the matrices when modules appear (the reader will easily consider the cases $n = 2$ and $n = 3$ oneself). Conjugation of the algebra with matrices of the form $I + sE_{3,2}$, $s \in \mathbb{C}$, (which are not upper-triangular) leave it upper-triangular, therefore we are not in the generic case here. All small blocks are of size 1 in the example.

Proof. 1^o. Factorize $\mathcal{A}$ as a linear space by the subspace $\mathbb{C}M$. Denote the factor by $\mathcal{B}$. Notice that if $(u, v) \neq (0, 0)$, then the Jordan normal form of the matrix $uK + vL + wM$ does not depend on w.

2^o. The factor $\mathcal{B}$ contains exactly two subspaces such that each non-zero matrix from them is conjugate to a Jordan matrix with two Jordan blocks 2×2. These are $V_1 = \mathbb{C}K$ and $V_2 = \mathbb{C}(\xi K + L)$, $\xi = -1/(a + 1)$ (to be checked directly). If two algebras $\mathcal{A}$ (obtained for $a = a'$ and $a = a''$) are equivalent, then the equivalence maps V_1 on V_1 or V_2, and V_2 on V_2 or V_1. If one fixes three different values of a, then there exists a couple of them – a', a'' – for which V_1 is mapped on V_1 and V_2 on V_2. This is what we presume from now on.

In all cases the space $\mathbb{C}M$ is mapped onto itself (the matrices from $\mathbb{C}M \setminus \{0\}$ are conjugate to Jordan nilpotent rank one matrices).

3^o. Therefore the flag of spaces $\mathcal{F}(a') = 0 \subset \mathbb{C}M \subset (\mathbb{C}M \oplus \mathbb{C}K(a')) \subset \mathcal{A}(a')$ is mapped onto $\mathcal{F}(a'')$ and the equivalence looks like this:

$$\begin{aligned} Q^{-1}MQ &= \eta M, && \eta \in \mathbb{C}^*, \\ Q^{-1}K(a')Q &= \delta K(a'') + \varepsilon M, && \delta \in \mathbb{C}^*, \quad \varepsilon \in \mathbb{C}, \\ Q^{-1}LQ &= \alpha L + \beta K(a'') + \gamma M, && \alpha \in \mathbb{C}^*, \quad \beta, \gamma \in \mathbb{C}, \end{aligned}$$

or, equivalently,

$$\begin{aligned} MQ &= \eta QM, \\ K(a')Q &= \delta QK(a'') + \varepsilon QM, \\ LQ &= \alpha QL + \beta QK(a'') + \gamma QM. \end{aligned}$$

The first of these relations implies that $Q_{2,1} = Q_{3,1} = Q_{4,1} = Q_{4,2} = Q_{4,3} = 0$. The second implies then that $Q_{2,3} = 0$.

4^o *One can assume that the following entries of Q are also 0: $Q_{1,2}$, $Q_{1,3}$, $Q_{1,4}$, $Q_{2,4}$ and $Q_{3,4}$.* Indeed, conjugation by matrices of the form $I + sE_{1,3}$ or $I + sE_{2,4}$, $s \in \mathbb{C}$, preserves the algebra $\mathcal{A}$ and the matrix Q is defined modulo the centralizer of $\mathcal{A}$.

Hence,

$$Q = \begin{pmatrix} b & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & f & d & 0 \\ 0 & 0 & 0 & g \end{pmatrix}.$$

The third relation (look at positions (1,3) and (2,4)) implies that $d = \alpha b$, $g = \alpha c$ (hence, $dc = bg$). The second relation implies (positions (1,2) and (3,4)) that $a'c = \delta b a''$, $g = \delta d$. Hence, $a'cd = a''bg$ which together with $dc = bg$ implies $a' = a''$ – a contradiction. The proposition is proved. □

4 The monodromy stratification of $(gl(n, \mathbb{C}))^p$ and of $(\mathrm{GL}(n, \mathbb{C}))^p$

Define a stratification of $(gl(n, \mathbb{C}))^p$ and of $(\mathrm{GL}(n, \mathbb{C}))^p$ (called a *monodromy stratification* because $(gl(n, \mathbb{C}))^p$ and $(\mathrm{GL}(n, \mathbb{C}))^p$ are regarded as spaces of p-tuples of residua of Fuchsian systems or of p-tuples of monodromy operators of regular systems).

Consider the set $\mathcal{R}$ of p-tuples of $(gl(n, \mathbb{C}))^p$ (for $(\mathrm{GL}(n, \mathbb{C}))^p$ the reasoning is similar) belonging (up to conjugacy) to a fixed reducibility pattern. This is a constructible subset of $(gl(n, \mathbb{C}))^p$. The strata of $\mathcal{R}$ are the connected components of the intersections $\mathcal{R} \cap \mathcal{B}_k$ where the constructible sets $\mathcal{B}_k$ are defined below. Hence, all strata are constructible connected subsets of $(gl(n, \mathbb{C}))^p$.

Define the subsets $\mathcal{B}_k$ for p-tuples of $(gl(n,\mathbb{C}))^p$ blocked as the reducibility pattern. (For arbitrary p-tuples they are defined as $Q^{-1}\mathcal{B}_k Q$ with a suitable matrix $Q \in \mathrm{GL}(n,\mathbb{C})$ and $\mathcal{B}_k$ defined for this special case.) Two p-tuples $(A_1,\dots,A_p)$ and $(A'_1,\dots,A'_p)$ belong to one and the same set $\mathcal{B}_k$ if and only if there exist matrices $Q_1,\dots,Q_{p+1} \in \mathrm{GL}(n,\mathbb{C})$ blocked as the reducibility pattern such that $Q_j^{-1}A_jQ_j = A'_j$, $j = 1,\dots,p+1$.

A stratum is called *irreducible* if it consists of one big and at the same time small block.

A stratum is called *special* if its reducibility pattern contains a special pair.

The set of all p-tuples of $(gl(n,\mathbb{C}))^p$ (any connected component of the set of p-tuples of $(\mathrm{GL}(n,\mathbb{C}))^p$) with given Jordan normal forms of the matrices $A_1,\dots,A_p$, $A_{p+1} = -A_1 - \dots - A_p$ (of the matrices $M_1,\dots,M_p, M_{p+1} = (M_1\dots M_p)^{-1}$) is called a *superstratum*. To understand why we add the requirement concerning the connected components in the case of $(\mathrm{GL}(n,\mathbb{C}))^p$ see the definition of arithmetic splitting at the end of the section.

Theorem 4.1. 1) *All irreducible strata of $(gl(n,\mathbb{C}))^p$ and all irreducible strata of $(\mathrm{GL}(n,\mathbb{C}))^p$ are locally smooth constructible sets.*

2) *If one of the matrices A_j (respectively, M_j) has distinct eigenvalues, then such an irreducible stratum is globally a connected and smooth constructible set.*

3) *The set of singular points of a superstratum coincides with the union of its special strata having a non-trivial centralizer.*

For strata of $(\mathrm{GL}(n,\mathbb{C}))^p$ parts 1) and 2) are proved in [17], see Theorem 2.2 there. For $(gl(n,\mathbb{C}))^p$ the proofs of 1) and 2) are carried out in the same way (in fact – more easily). The smoothness is derived from the solvability of the equation

$$\sum_{j=1}^{p+1}[A_j, X_j] = S, \quad \mathrm{tr}S = 0 \tag{13}$$

with respect to the unknown matrices $X_j \in gl(n,\mathbb{C})$. One can use the fact that the mapping $(X_1,\dots,X_{p+1}) \mapsto \sum_{j=1}^{p+1}[A_j, X_j]$ is surjective if and only if the centralizer of the $(p+1)$-tuple of matrices A_j is trivial (which is the case if it is irreducible – Schur's lemma).

Statement 3) of the theorem is proved in [17], p. 267, see 7^o there.

The theorem is proved also in the case when in its formulation a stratum is replaced by a variety consisting of all $(p+1)$-tuples of matrices A_j or M_j from given conjugacy classes, see [21].

Definition 4.2. We call *arithmetic splitting* the following phenomenon in $(\mathrm{GL}(n,\mathbb{C}))^p$. If the greatest common divisor l of all the multiplicities of the eigenvalues of the matrices $M_1,\dots,M_{p+1}$ is greater than 1, then there are exactly l generic irreducible strata, i.e. such that the characteristic and the minimal polynomials of M_j coincide for $j = 1,\dots,p+1$ (except in the case $p = n = 2$) with such multiplicities of the eigenvalues. Indeed, denote the eigenvalues and their multiplicities by $\lambda_1, \ldots, \lambda_s$, $lm_1, \ldots, lm_s$. Then the equation

$$(\lambda_1^{m_1} \ldots \lambda_s^{m_s})^l = 1$$

implies that one of the l equations holds:

$$\lambda_1^{m_1} \ldots \lambda_s^{m_s} = \omega_j$$

where ω_j are all the roots of unity of order l. Each of the last equations defines an irreducible non-singular variety in the space of eigenvalues (this is left for the reader to prove). Each of these varieties corresponds to a different generic stratum.

For $p = n = 2$ there are (resp. there are not) irreducible monodromy groups in which the operators M_1, M_2, M_3 are conjugate to Jordan blocks 2×2 with eigenvalues $(1, 1, -1)$ (resp. with eigenvalues $(1, 1, 1)$).

An example of arithmetic splitting for $n = 3$, $p = 2$ is given in [17], p. 263.

5 Sufficient conditions for reducible groups to be realized by reduced Fuchsian systems

In this section we consider regular systems with reducible monodromy groups. Using Lemma 2.5, we consider systems in block upper-triangular form only, the block structure being the same as the one of the reducibility pattern of the monodromy group. Our aim is to find a sufficient condition for the equivalence of the regular system to a Fuchsian one, of the same block upper-triangular form. To this end we introduce the following

Definition 5.1. Call a *special admissible set of integers* (*SASI*) any set of integers $\varphi_{k,j}$, $k = 1, \ldots, n$; $j = 1, \ldots, p+1$ (see Theorem 2.7) with the following properties:

i) There exists an integer l, $1 \leq l \leq p+1$ such that for $j \neq l$ one has $\varphi_{\mu,j} = \varphi_{\nu,j}$ whenever $\varphi_{\mu,j}$ and $\varphi_{\nu,j}$ correspond to one and the same eigenvalue of the operator M_j.

ii) Suppose that $\varphi_{k_1,l}, \ldots, \varphi_{k_s,l}$ are all the integers $\varphi_{k,l}$ corresponding to one and the same eigenvalue of the operator M_l ($k_1 > k_2 > \cdots > k_s$) which is assumed to be in upper-triangular form, see Subsection 2.5. Then one has

$$\varphi_{k_i,l} - \varphi_{k_{i+1},l} \geq (n-1)(p+3)$$

(for all such sets $\varphi_{k_1,l}, \ldots, \varphi_{k_s,l}$)

iii) For every small diagonal block one has

$$\sum_k \sum_{j=1}^{p+1} (\beta_{k,j} + \varphi_{k,j}) = 0$$

Theorem 5.2. *If for a reducible monodromy group there exists a SASI, then the group is the monodromy group of a Fuchsian system for which the values of $\varphi_{k,j}$ for $j \neq l$ are the ones from the SASI and the values of $\varphi_{k,l}$ differ from the corresponding values from the SASI by no more than $(n-1)(p+3)/2$. This system is in the same block upper-triangular form as the one of the reducibility pattern of the monodromy group.*

Remark 5.3. If the reducibility pattern contains more than one big block, then it is sufficient to find a SASI for the different big blocks separately; the indices l for them need not be the same.

Corollary 5.4. *Suppose that the reducibility pattern of the monodromy group is $\left(\begin{smallmatrix} P & Q \\ 0 & R \end{smallmatrix}\right)$ (i.e. the group is a semi-direct but not direct sum). Suppose that at least one of the monodromy operators M_j has $r \geq 2$ eigenvalues. Let $m_1, \dots, m_r$ be the multiplicities of the eigenvalues as eigenvalues of the P-block and $q_1, \dots, q_r$ be their multiplicities as eigenvalues of the R-block. If the system of equations and inequalities*

$$\begin{aligned} m_1\sigma_1 + \cdots + m_r\sigma_r &= 0, \\ q_1\tau_1 + \cdots + q_r\tau_r &= 0, \\ \sigma_1 > \tau_1, \dots, \sigma_r &> \tau_r \end{aligned} \tag{14}$$

has a real solution, then the group in question is the monodromy group of a Fuchsian system, in block upper-triangular form, with the same sizes of the blocks as the ones of the reducibility pattern.

This system has a solution if and only if the vectors $(m_1, \dots, m_r)$ and $(q_1, \dots, q_r)$ are not collinear (i.e. the multiplicities are not proportional). In particular, if $r = 2$, then such a solution exists if and only if $m_1q_2 - m_2q_1 \neq 0$.

Corollary 5.4 is proved at the end of the section, after the proof of the theorem. It is clear that the solution (σ, τ) (if it exists) can be chosen rational. In a similar way can be proved

Corollary 5.5. *Suppose that the reducibility pattern of the monodromy group of a regular system is*

$$\begin{pmatrix} P_1 & Q_1 & \dots & R_1 \\ 0 & P_2 & \dots & R_2 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & P_s \end{pmatrix}$$

and that the multiplicities of the eigenvalues of the monodromy operator M_j as eigenvalues of the P_k-block are $m_1^k, \dots, m_r^k$. If for at least one j, $1 \leq j \leq p+1$, the system of equations and inequalities

$$\begin{aligned} m_1\sigma_1^1 + \cdots + m_r^1\sigma_r^1 &= 0 \\ &\vdots \\ m_1^s\sigma_1^s + \cdots + m_r^s\sigma_r^s &= 0 \\ \sigma_k^1 > \sigma_k^2 > \cdots > \sigma_k^s, &\quad k = 1, \dots, r \end{aligned}$$

has a real solution, then the regular system in question is equivalent to a Fuchsian one in the same block upper-triangular form. If the reducibility pattern contains more than one big block, then if this criterion is fulfilled by each big block (*possibly, for different indices* j)*, then the corollary is also true. The solution to the system of equations and inequalities* (*if it exists*) *can be chosen rational.*

Corollary 5.6. *If one of the operators* $M_1, \ldots, M_{p+1}$ *contains only one Jordan block of size* 2 *and* $(n-2)$ *blocks of size* 1 *in its Jordan normal form, then the system is equivalent to a Fuchsian one* (*regardless of the reducibility and choice of the positions of the poles* a_j)*.*

Proof of Theorem 5.2. We use the same ideas as the ones from the proof of Theorem 2.12:

1) Assume that a regular system in block upper-triangular form is given which has the monodromy group in question. Create an apparent singularity at some point and change all the integers $\varphi_{k,j}$ to the ones from the SASI, see Subsection 2.7, parts (B), (C). These procedures can be chosen preserving the initial block upper-triangular form.

2) Let φ_k^0 be the integers $\varphi_{k,j}$ corresponding to the apparent singularity. Similarly to Subsection 2.7, parts (D), (E), (F), we change these numbers so that one has

$$0 \leq \varphi_k^0 - \varphi_{k+1}^0 \leq p+1.$$

Moreover, we can make this singularity Fuchsian and these procedures can be chosen preserving the initial block upper-triangular structure as well. Indeed, we can perform these procedures for every small diagonal block. After this it is possible that there will appear poles at a^0 in the blocks above the diagonal of the U_j-matrices. They can be removed by an equivalence in block upper-triangular form (we leave the details for the reader).

3) The fact that after 2) the system is Fuchsian and in block upper-triangular form implies that for every small diagonal block one has

$$\sum_k \sum_{j=1}^{p+1} (\beta_{k,j} + \varphi_{k,j}) + \sum_k \varphi_k^0 = 0.$$

Hence, $\sum_k \varphi_k^0 = 0$ for every small block. This can be proved in the same way as it is proved for a Fuchsian system (not for a diagonal block); note that the diagonal blocks of the system are Fuchsian systems themselves.

4) We remove the apparent singularity as in Subsection 2.7, part (G). This can be done by a procedure preserving the block upper-triangular structure. One can easily show (using 2) and $\sum_k \varphi_k^0 = 0$) that the integers $\varphi_{k,l}$ change by $\psi_{k,l} \in \mathbb{Z}$, where

$$-(n-1)(p+3)/2 \leq \psi_{k,l} \leq (n-1)(p+3)/2$$

Hence, the final values of $\varphi_{k,l}$ will satisfy the conditions of the theorem and condition (7) from Theorem 2.7. □

Proof of Corollary 5.6. 1^o. Let $\varphi_{k_1,p+1}, \varphi_{k_2,p+1}$ be the integers corresponding to the only Jordan block of size 2 of the operator M_{p+1}. If they correspond to one and the same small diagonal block or to two different small diagonal blocks at least one of which is of size > 1, then the existence of a SASI is evident (one can put $\varphi_{k,j} = 0$ for $j \neq p+1$).

2^o. If they correspond to two small blocks of size 1, then one must first fix $\varphi_{k_1,p+1} = \varphi_{k_2,p+1}$ and then choose the values of $\varphi_{k,j}$, $j \neq p+1$, so that

1) $\varphi_{\mu,j} = \varphi_{\nu,j}$ whenever $\varphi_{\mu,j}, \varphi_{\nu,j}$ correspond to one and the same eigenvalue of M_j, for every $j \neq p+1$;

2) $\sum_j (\beta_{k_1,j} + \varphi_{k_1,j}) = 0$, $\sum_j (\beta_{k_2,j} + \varphi_{k_2,j}) = 0$.

3^o. Note that condition 1) gives two possible cases – either for all j one has $\varphi_{k_1,j} = \varphi_{k_2,j}$, i.e. for all j the numbers $\varphi_{k_1,j}$, $\varphi_{k_2,j}$ of the same small blocks as $\varphi_{k_1,p+1} = \varphi_{k_2,p+1}$ correspond to one and the same eigenvalue of M_j and, hence, are equal, or for at least one $j \neq p+1$ they correspond to different eigenvalues of M_j. In both cases it is possible to choose the numbers $\varphi_{k,j}$ satisfying conditions 1) and 2).

4^o. After this any values of $\varphi_{k,p+1}$ for $k \neq k_1, k_2$ satisfying iii) of the definition of a SASI will complete the set to a SASI. Hence, Corollary 5.6 can be made more exact: *A regular system satisfying the conditions of the corollary is equivalent to a Fuchsian one in block upper-triangular form, the same as the reducibility pattern of the monodromy group of the system.* □

Proof of Corollary 5.4. 1^o. Let M_{p+1} have at least two eigenvalues. We use Theorem 5.2 and prove the existence of a SASI. Put $\varphi_{k,j} = 0$ for $j = 1, \dots, p$. For the P- and R-blocks considered separately fix the integers $\varphi_{k,p+1}$ such that $\sum_k \sum_{j=1}^{p+1} (\beta_{k,j} + \varphi_{k,j}) = 0$ (the sums are taken and the equations are considered over the P- and R-blocks separately) and $\varphi_{k_j,p+1} \geq \varphi_{k_{j+1},p+1} + (n-1)(p+3)$ if $\varphi_{k_j,p+1}$ are all the numbers $\varphi_{k,j}$ corresponding to one Jordan block of the P- (of the R-) block.

2^o. Let $(\sigma_1, \dots, \sigma_r, \tau_1, \dots, \tau_r)$ be an integer solution to the system from the corollary (if there exists a rational, then there exists an integer solution; if (σ, τ) is a solution, then $(b\sigma, b\tau)$ is also a solution, for any $b > 0$). If the integer $\varphi_{k,p+1}$ corresponds to the same eigenvalue of M_{p+1} to which correspond the numbers σ_i, τ_i, then we set $i = \kappa(k)$. Hence, for $b \in \mathbb{N}$ sufficiently large the set of integers

$$\varphi_{k,j} = 0,\ j = 1, \dots, p; \quad \varphi_{k,p+1} + b\sigma_{\kappa(k)} \text{ (for the } P\text{-block of } M_{p+1}),$$

$$\varphi_{k,p+1} + b\tau_{\kappa(k)} \text{ (for the R-block of } M_{p+1})$$

will be a SASI, due to $\sigma_i > \tau_i$.

3^o. Let there exist a pair (i, j) for which $m_i q_j - m_j q_i \neq 0$. Assume that $i = 1$, $j = 2$. Solve first the system

$$\begin{aligned} &m_1\sigma_1 + m_2\sigma_2 = 0, \quad q_1\tau_1 + q_2\tau_2 = 0, \\ &\sigma_1 > \tau_1,\ \sigma_2 > \tau_2. \end{aligned} \tag{15}$$

If $(m_1/m_2) > (q_1/q_2)$, then one can choose $\tau_1 < \sigma_1 < 0$ such that

$$(m_2q_1/m_1q_2) < |\sigma_1|/|\tau_1| < 1.$$

Hence,

$$\sigma_2 = -m_1\sigma_1/m_2 > -q_1\tau_1/q_2 = \tau_2$$

If $(m_1/m_2) < (q_1/q_2)$, then one can choose $0 < \tau_1 < \sigma_1$ such that

$$(m_2q_1/m_1q_2) > \sigma_1/\tau_1 > 1.$$

Hence, again $\sigma_2 = -m_1\sigma_1/m_2 > -q_1\tau_1/q_2 = \tau_2$.

Denote by $(\sigma_1^0, \sigma_2^0, \tau_1^0, \tau_2^0)$ a solution to system (15). Fix σ_j, τ_j such that $\sigma_j > \tau_j$, $j = 3,\ldots,r$. Fix a solution (μ, ν) to the system

$$\begin{aligned} m_1\mu + m_2\nu + m_3\sigma_3 + \cdots + m_r\sigma_r &= 0 \\ q_1\mu + q_2\nu + q_3\tau_3 + \cdots + q_r\tau_r &= 0 \end{aligned}$$

For $\alpha > 0$ sufficiently large the numbers

$$\begin{aligned} &\sigma_1 = \mu + \alpha\sigma_1^0, \quad \sigma_2 = \nu + \alpha, \sigma_2^0, \quad \sigma_3, \ldots, \sigma_r, \\ &\tau_1 = \mu + \alpha\tau_1^0, \quad \tau_2 = \nu + \alpha\tau_2^0, \quad \tau_3, \ldots, \tau_r \end{aligned}$$

provide a solution to system (14). This solution can be chosen rational.

4^o. If $m_1/q_1 = \cdots = m_r/q_r$, then system (14) has no solution (the proportionality implies that if $\sigma_i > \tau_i$, then for some j one has $\sigma_j < \tau_j$). The corollary is proved. □

6 The codimension of the negative answer to the Riemann–Hilbert problem

In this section we treat the following question: for fixed poles what is the codimension in $(\mathrm{GL}(n,\mathbb{C}))^p$ of the groups $\{M_1, \ldots, M_p\}$ which are not monodromy groups of Fuchsian systems with the given set of poles and no other singularities?

6.1 The main results

Definition 6.1. Call a stratum of $(\mathrm{GL}(n,\mathbb{C}))^p$ *good* if for any choice of the points $a_1, \ldots, a_{p+1}$ and for any choice of a group $\{M_1, \ldots, M_p\}$ belonging to the stratum there exists a Fuchsian system on $\mathbb{C}P^1$ with poles at a_j and only there for which this group is its monodromy group. In the opposite case the stratum is called *bad*.

Example 6.2. Every irreducible stratum is good, see Theorem 2.2. If one of the monodromy operators is diagonalizable, then the stratum is good (regardless of reducibility); this follows from Plemelj's wrong proof in [26] of the Riemann–Hilbert problem, see [3]. Corollary 5.6 gives further examples of good strata.

Definition 6.3. For $p \geq 2$ and $n \geq 3$ call *main bad strata* those strata for which

1) the reducibility pattern is $\left(\begin{smallmatrix} a & N \\ 0 & P \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} P & N \\ 0 & a \end{smallmatrix}\right)$, where P is $(n-1) \times (n-1)$;

2) each of the matrices $M_1, \ldots, M_{p+1}$ is conjugate to one Jordan block $n \times n$.

Theorem 6.4. *The dimension of each of the main bad strata is equal to* $p(n^2 - 2n + 2) + 1$.

Proof. 1^o. The dimension of the groups belonging to one of the main bad strata in which M_j are already in block upper-triangular form is equal to

$$\kappa = \kappa_1 + \kappa_2, \quad \kappa_1 = p(n-1)^2 - (p+1)(n-2), \quad \kappa_2 = p(n-1).$$

Indeed,

1) κ_1 is the dimension of the "restriction of the matrices M_j to the P-block" (the term $p(n-1)^2$ equals the dimension of the space of p-tuples $P_1, \ldots, P_p$ and we subtract $(p+1)(n-2)$, because $n-2$ is the number of equalities between eigenvalues of a given matrix P_j);

2) κ_2 is the dimension of the "restriction to the N-block" (once the P-block is defined, the number a is defined, too – it is an eigenvalue – and for almost every choice of the N-blocks the matrices M_j will be conjugate to Jordan blocks $n \times n$ with eigenvalue a).

2^o. Further we consider only the case of the reducibility pattern $\left(\begin{smallmatrix} a & N \\ 0 & P \end{smallmatrix}\right)$, the one of the reducibility pattern $\left(\begin{smallmatrix} P & N \\ 0 & a \end{smallmatrix}\right)$ is considered by analogy. The set of groups described in 1^o is invariant under conjugation with matrices of the kind $S = \left(\begin{smallmatrix} a & N \\ 0 & P \end{smallmatrix}\right)$. Every non-degenerate matrix G whose entry $G_{1,1}$ is non-zero can be represented in a unique way as a product TS, where S is as above and $T = \left(\begin{smallmatrix} 1 & 0 \\ T' & I \end{smallmatrix}\right)$, where I is $(n-1) \times (n-1)$.

Every non-degenerate matrix G with $G_{1,1} = 0$ and $G_{2,1} \neq 0$ can be represented as QS with

$$Q = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & Q' & I \end{pmatrix},$$

I being $(n-2) \times (n-2)$. If $G_{1,1} = G_{2,1} = 0$, $G_{3,1} \neq 0$, then $G = RS$ with

$$R = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & R' & I \end{pmatrix}$$

etc. The matrices Q', R' etc. contain less than $n-1$ parameters.

Hence, to obtain the final answer, we must add $(n-1)$ (the dimension of the space of matrices T), i.e. we must take into account the possibility to conjugate with matrices of this kind. The theorem is proved. □

Denote for $p \geq 3, n \geq 3$ by Σ any of the main bad strata and by Σ_0 the subset (for $a_1, \dots, a_{p+1}$ fixed) of groups $\{M_1, \dots, M_p\} \in \Sigma$ which are not monodromy groups of Fuchsian systems on $\mathbb{C}P^1$ with poles at $a_1, \dots, a_{p+1}$ (and only there).

Theorem 6.5. *If* $n \geq 3,\ p \geq 3$, *then for any fixed* $(a_1, \dots, a_{p+1})$ *the set* Σ_0 *is* (*locally*) *a proper analytic submanifold of* Σ *and there exist points where its codimension is equal to* 1. *Hence,* $\operatorname{codim}_{(\mathrm{GL}(n,\mathbb{C}))^p} \Sigma_0 = 2p(n-1)$.

The theorem is proved in Subsection 6.2. It would be interesting to compare this result with a similar result from [1], p. 129 given in a somewhat different context.

Let Σ' be a bad, but not a main bad stratum.

Theorem 6.6. *For* $p \geq 3, n \geq 7$ *one has* $\dim \Sigma' \leq \dim \Sigma_0$.

The cases $n = 4, 5, 6$ are briefly considered in Subsection 6.3.

Remarks 6.7. 1) The theorem justifies the definition "main bad stratum" and shows that for $n \geq 7,\ p \geq 3$ the codimension of the set of groups which are not monodromy groups of Fuchsian systems with prescribed poles is equal to $2p(n-1)$. For $n = 4$ and 6 (and only for these values of n) there are strata different from the main bad ones on which the codimension $2p(n-1)$ of the subset of bad groups is also attained, but it is never smaller than $2p(n-1)$.

2) A. A. Bolibrukh has given examples (see [7] or [1], p. 105, Example 5.3.1) of monodromy groups which are bad regardless of the choice of the poles a_j due to the arithmetics of the eigenvalues of the monodromy matrices. The above theorem shows that nevertheless the codimension in $(\mathrm{GL}(n,\mathbb{C}))^p$ of their set is greater than the one of Σ_0.

3) Theorem 6.5 remains true for $p = 2$ as well provided that n is large enough. In this case the definition of Σ_0 does not depend on the position of the poles. We do not give the proof of this fact but just an illustration for $p = 2, n = 4$, see Example 6.8 in Subsection 6.4.

4) For $p = 2,\ n = 3$ the set Σ_0 is empty (this follows from the results of the paper [5]).

5) For $p = 2$ and for certain small values of n the smallest codimension (of the subset of bad monodromy groups in $(\mathrm{GL}(n,\mathbb{C}))^p$) is attained not on Σ_0, see Example 6.9 in Subsection 6.4.

6) For $n = 3$ the main bad strata are the only bad ones, see [5]; hence, Theorem 6.5 gives the codimension of the "bad" groups for $n = 3$ as well.

Proof of Theorem 6.6. 1^o. Suppose first (in 1^o–3^o) that the reducibility pattern of a stratum consists of one big and two small blocks: $\left(\begin{smallmatrix} P & Q \\ 0 & R \end{smallmatrix}\right)$, Q is $(n-k) \times k$. We show (in 2^o) that if the stratum is bad, then its codimension is at least $(p-1)k(n-k) + (p+1)\max(k, n-k) - 1$. For this purpose we use Corollary 5.4.

2^o. The dimension of the stratum is not greater than

$$\kappa = \kappa_1 - \kappa_2 + 1, \quad \kappa_2 = (p+1)\max(k, n-k)$$

$$\kappa_1 = p(k^2 + k(n-k) + (n-k)^2) + k(n-k) = p(\dim P + \dim Q + \dim R) + k(n-k).$$

The term κ_1 equals the dimension of the set of p-tuples of matrices M_j which are semi-direct sums with dimension of the subspace and factorspace equal to $(n-k)$ and k. The term $k(n-k)$ in κ_1 stands for the possibility to conjugate such a block upper-triangular p-tuple with matrices $\left(\begin{smallmatrix} I & 0 \\ T & I \end{smallmatrix}\right)$, T being $k \times (n-k)$, similarly to the proof of Theorem 6.4.

The term κ_2 is equal to the number of equalities between eigenvalues of one and the same matrix M_j. Note that $\min(k, n-k)$ is the greatest possible number of different eigenvalues of M_j, $j = 1, \ldots, p+1$ (if the number of eigenvalues is greater than $\min(k, n-k)$, then there exists a SASI, see the previous section, and the stratum is good). Fixing an eigenvalue is equivalent to imposing an analytic condition. The term 1 in κ shows that if in the last operator one has fixed all but one of the eigenvalues of the $P-$block (or of the $R-$block, if its dimension is the smaller of the two), then the last eigenvalue is determined from the condition $M_1 \ldots M_{p+1} = I$.

Hence, for $p \geq 3$, $n \geq 3$ the codimension of a bad stratum in $(\mathrm{GL}(n,\mathbb{C}))^p$ is at least

$$pn^2 - p(k^2 + (n-k)^2 + k(n-k)) + (p+1)\max(k, n-k) - k(n-k) - 1$$

$$= (p-1)k(n-k) + (p+1)\max(k, n-k) - 1.$$

3^o. We prove here that for $n \geq 7$, $p \geq 3$ one has

$$2p(n-1) = \mathrm{codim}_{(\mathrm{GL}(n,\mathbb{C}))^p} \Sigma_0 \leq (p-1)k(n-k) + (p+1)\max(k, n-k) - 1$$

$$= \sigma(p, n, k) \leq \mathrm{codim}_{(\mathrm{GL}(n,\mathbb{C}))^p} \Sigma'.$$

Note that for (p, n) fixed, σ is a quadratic polynomial in k for $k \in [2, [n/2]]$ or for $k \in [n - [n/2], n-2]$; here [.] denotes "the integer part of". We do not consider the cases $k = 1$ and $k = n-1$ – they give either the main bad strata or good strata, see Corollary 5.4. Let $k \in [2, [n/2]]$ (i.e. , $k \geq n-k$; the case $k \in [n - [n/2], n-2]$ is considered by analogy). Then σ is minimal either for $k = 2$ or for $k = [n/2]$ and we must check that

$$2p(n-1) \leq (p-1)2(n-2) + (p+1)(n-2) - 1 = (3p-1)(n-2) - 1$$

(for $k = 2$; this inequality is true for $p \geq 3, n \geq 6$) or

$$2p(n-1) \leq (p-1)[n/2](n - [n/2]) + (p+1)(n - [n/2]) - 1$$

(for $k = [n/2]$; this inequality has not to be checked for $n = 4$ and 5; it is true for $n \geq 7$, $p \geq 3$; it is not true for $n = 6$, $p = 3$; for $p \geq 4$, $n = 6$ it is true again).

4^o. Suppose now that the reducibility pattern of a stratum consists of one big and three small blocks. Let their sizes be k, l, $n-k-l$. Similarly to the proof of Theorem 6.4 we find that the set of groups having this reducibility pattern has codimension

$$\tau' = (p-1)(kl + (k+l)(n-k-l))$$

in $(\mathrm{GL}(n,\mathbb{C}))^p$. According to Corollary 5.6, the codimension of the bad strata of this reducibility pattern is at least $\tau' + 2(p+1)$ (each operator M_j must satisfy a codimension 2 condition – not to be diagonalizable and not to have just one Jordan block 2×2 in which cases the stratum is good, see the previous section).

5^o. The number τ' is minimal in the case when two of the numbers $k, l, n-k-l$ are equal to 1; this case will be considered in 6^o. From the other cases τ' is minimal if one of these numbers is equal to 1 and another one to 2. In this case one has

$$\tau' + 2(p+1) = (p-1)(2 + 3(n-3)) + 2(p+1) = (p-1)(3n-7) + 2(p+1)$$

Consider the inequality

$$\tau' + 2(p+1) \geq 2p(n-1),$$

or, equivalently,

$$(p-1)(3n-7) + 2(p+1) \geq 2p(n-1) \quad \text{or} \quad (n-3)(p-3) \geq 0.$$

It is true for $p \geq 3, n \geq 3$.

6^o. In the exceptional case $k = l = 1$ only for $n \geq 4$ the stratum can be bad, see [5]. For $n \geq 4$ a stratum is bad only if each eigenvalue of the block $(n-2) \times (n-2)$ is eigenvalue of one of the blocks 1×1, otherwise one easily constructs a SASI, see the previous section. This makes $\kappa'' = (p+1)(n-2)$ equalities between the eigenvalues. Hence, the codimension of the bad strata is at least $\kappa' + \kappa'' - 1$,

$$\kappa' = (p-1)(1 + 2(n-2)), \quad \kappa' + \kappa'' - 1 = 3pn - n - 5p$$

(κ' is responsible for the reducibility and κ'' – for the equalities between the eigenvalues). One has

$$3pn - n - 5p - 2p(n-1) = pn - n - 3p = (p-1)(n-3) - 3 \geq 0$$

(the last inequality is true for $p \geq 3, n \geq 5$).

7^o. If the reducibility pattern of a stratum contains one big and at least 4 small blocks, then its codimension in $(\mathrm{GL}(n,\mathbb{C}))^p$ is minimal if the sizes of the small blocks are $1, \ldots, 1, n-k$; $(k+1)$ being their number. The codimension is at least

$$\tau = (p-1)(k(n-k) + k(k-1)/2) + 2(p+1)$$

which is minimal for $k = 4$ or $k = n$. In these cases one has respectively

$$\tau = (p-1)(4n-10) + 2(p+1) > 2p(n-1)$$

(true for $n \geq 5$, $p \geq 3$) and

$$\tau = (p-1)n(n-1)/2 + 2(p+1) > 2p(n-1)$$

(true for $n \geq 5$, $p \geq 3$).

8^o. We let the reader prove that if the reducibility pattern of a bad stratum contains at least two big blocks, then its codimension is bigger than $2p(n-1)$; the proof is similar to the one of the cases considered and we omit it. □

6.2 Proof of Theorem 6.5

1^o. Describe first the set Σ_0. We use Bolibrukh's conception of a Fuchsian weight here, see [5]. Let the reducibility pattern of the stratum be $\left(\begin{smallmatrix} a & P \\ 0 & Q \end{smallmatrix}\right)$, Q being $(n-1)\times(n-1)$. If the system is Fuchsian, then one has $\sum_{k=1}^{n}\sum_{j=1}^{p+1}\varphi_{k,j} + n\sum_{j=1}^{p+1}\beta_j = 0$ ($\beta_{k,j}$ do not depend on k). One also has (see [5])

$$0 \geq \sum_{j=1}^{p+1}(\varphi_{1,j} + \beta_j) \in \mathbb{Z}. \tag{16}$$

If this inequality is strict, then for some $k > 1$ and some j, $1 \leq j \leq p+1$ one must have $\varphi_{k,j} > \varphi_{1,j}$, i.e. (7) does not hold (we assume that (6) is fulfilled) because we also have (8). But then the system cannot be Fuchsian at a_j which is a contradiction. Hence, one must have

$$\sum_{j=1}^{p+1}(\varphi_{1,j} + \beta_j) = 0. \tag{17}$$

This implies, see [7], that the system is equivalent to a block upper-triangular Fuchsian one (denoted by BUTF). The Q-block of BUTF itself is a Fuchsian system and one has

$$\sum_{k=2}^{n-1}\sum_{j=1}^{p+1}\varphi_{k,j} + (n-1)\sum_{j=1}^{p+1}\beta_j = 0. \tag{18}$$

It is easy to see that the system is Fuchsian if and only if

$$\varphi_{1,j} = \varphi_{2,j} = \cdots = \varphi_{n,j} \tag{19}$$

for $j = 1, \ldots, p+1$. On the other hand, a regular system with a monodromy group belonging to Σ is equivalent to a block upper-triangular one (denoted by (S)). If the equalities

$$\varphi_{2,j} = \cdots = \varphi_{n,j}, \quad j = 1, \ldots, p+1 \tag{20}$$

for (S) hold, then the Q-block of (S) is itself a Fuchsian system and the a-block is one as well. Hence, one has equalities (18) and (17) (the sum of the residua of a Fuchsian system on $\mathbb{C}P^1$ is zero). This implies that (19) is true (one can always assume that $\varphi_{k,j} = 0$ for $j \neq p+1$), i.e. system (S) is Fuchsian.

Conclusion. Let the poles a_j be fixed. Then a regular system with a monodromy group belonging to Σ is not equivalent to a Fuchsian one if and only if the Q-block of the monodromy group is not the monodromy group of a Fuchsian system (of dimension $n-1$) for which one has (20); see [7] as well. Denote the set of such $(n-1)\times(n-1)$

Fuchsian systems by $\Sigma^*_{0,n-1}$. Call it *the set of Fuchsian systems of non-zero Fuchsian weight*. Unlike Bolibrukh, in defining the Fuchsian weight we restrict ourselves to the case when all monodromy operators M_j are conjugate to Jordan blocks $(n-1)\times(n-1)$.

2^o. Consider the stratum $\tilde{\Sigma}$ of all irreducible monodromy groups of regular systems with fixed poles in which each of the operators $M_1, \ldots, M_{p+1}$ is conjugate to one Jordan block. They can be realized by Fuchsian systems with $\varphi_{k,j} = 0$ for $j = 1, \ldots, p$; $k = 1, \ldots, n$ and $s_1 \leq \varphi_{k,p+1} \leq s_2$ for some $s_1, s_2 \in \mathbb{Z}$. This can be proved in the same way as Theorem 2.12. Fix the numbers $\beta_{k,j}$, $\mathrm{Re}\beta_{k,j} \in [0, 1)$ not depending on k and satisfying the condition $\sum_{j=1}^{p+1} \beta_{k,j} \in \mathbb{Z}$. Then there exists a finite set $\mathcal{S}$ of n-tuples $\vec{\varphi} = (\varphi_{1,p+1}, \ldots, \varphi_{n,p+1})$ such that $s_2 > \varphi_{1,p+1} \geq \cdots \geq \varphi_{n,p+1} > s_1$ and there holds equality (8).

Introduce a partial ordering on the set $\mathcal{S}$: set $\vec{\varphi}'' - \vec{\varphi}' := (\varphi_1, \ldots, \varphi_n)$. One has $\vec{\varphi}' \prec \vec{\varphi}''$ if and only if there exists an integer $1 \leq j_1 \leq n$ such that $\varphi_j = 0$ for $j \in [1, j_1 - 1]$ and $\varphi_j > 0$ for $j = j_1$. Obviously, there exists a unique least entry $\vec{\varphi}^0 \in \mathcal{S}$, with $\varphi_{1,p+1} = \cdots = \varphi_{n,p+1}$. The next entry $\vec{\varphi}^1$ is also unique – for it one has $\varphi_{1,p+1} - 1 = \varphi_{2,p+1} = \cdots = \varphi_{n-1p+1} = \varphi_{n,p+1} + 1$.

3^o. Consider the mapping $\Delta\colon \{A_j\}_{j=1}^p \mapsto \{M_j\}_{j=1}^p$ with $A_{p+1} := -A_1 - \cdots - A_p$ fixed and the entries of the residua $\{A_j\}_{j=1}^{p-1}$ being considered as independent variables denoted by α. Put

$$A_{p+1} = \begin{pmatrix} \lambda+1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & \lambda & 1 & \ldots & 0 & 0 & 0 \\ 0 & 0 & \lambda & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \lambda & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & \lambda & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & \lambda-1 \end{pmatrix}.$$

Our aim is to perform a transformation

$$X \mapsto \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ h/(t-a_{p+1}) & 0 & \ldots & 0 & 1 \end{pmatrix} X, \quad h = h(\alpha, a_1, \ldots, a_{p+1})$$

after which the matrix-residuum A_{p+1} will have only one eigenvalue (namely, λ) and the system will remain Fuchsian. One can show that this is the form of the equivalence (2) (modulo equivalences with constant matrices W) which makes all eigenvalues of A_{p+1} equal without changing the ones of $A_1, \ldots, A_p$. A direct computation shows that

$$h = 1/\Big(\sum_{j=1}^{p} \alpha^j_{1n}/(a_j - a_{p+1})\Big) \tag{21}$$

where α^j_{1n} is the entry in the first row and n-th column of A_j. Hence, on the hyperplane $\{h = 0\}$ in the space $\{A_1, \dots, A_p\}$ this transformation is not defined. Put $\lambda = \sigma + \varphi$, $\mathrm{Re}\sigma \in [0, 1), \varphi \in \mathbb{Z}$. Then the integers $\varphi_{k,p+1}$ of the system before the transformation are equal to $(\varphi + 1, \varphi, \dots, \varphi, \varphi - 1)$. The transformation makes them all equal to φ.

After the transformation the matrix-residuum A_{p+1} becomes equal to

$$\begin{pmatrix} \lambda & 0 & 0 & \dots & 0 & 0 & 0 \\ d_2 & \lambda & 1 & \dots & 0 & 0 & 0 \\ d_3 & 0 & \lambda & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ d_{n-2} & 0 & 0 & \dots & \lambda & 1 & 0 \\ d_{n-1} & 0 & 0 & \dots & 0 & \lambda & 0 \\ c_1 & c_2 & c_3 & \dots & c_{n-2} & c_{n-1} & \lambda \end{pmatrix}, \quad c_k = f_k/h, \ d_k = g_k/h$$

where f_k (resp. g_k) are linear functions of α^ν_{1k} (resp. of α^ν_{kn}), $\nu = 1, \dots, p$; $k = 1, \dots, n-1$ with coefficients depending on a_j, $j = 1, \dots, p+1$. There exists a proper analytic subset of the space $\{A_1, \dots, A_p\}$ such that for every choice of α from its complement

1) the above transformation is defined;

2) the new matrix-residuum A_{p+1} is conjugate to one Jordan block $n \times n$ with eigenvalue λ;

3) if α belongs to the subset in question, then at least one of 1) and 2) is not true.

4^o^. Show by explicit construction that the intersection $\tilde{\Sigma} \cap \{h = 0\}$ is non-empty. Consider the Fuchsian system $\dot{X} = \big[\sum_{j=1}^{p+1} A_j/(t - a_j)\big]X$, $A_1 + \dots + A_{p+1} = 0$ where

$$A_{p+1} = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & -1 \end{pmatrix},$$

$$A_j = D^{-1}(\gamma_j)C(d_j, b_j, c_j)D(\gamma_j), \quad j = 1, \dots, p$$

$$D(\gamma) = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ -\gamma & 0 & \dots & 0 & 1 \end{pmatrix},$$

$$C(a,b,c)=\begin{pmatrix}0&c&0&0&\dots&0&0&d\\0&0&b&0&\dots&0&0&0\\0&0&0&b&\dots&0&0&0\\0&0&0&0&\dots&0&0&0\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots\\0&0&0&0&\dots&0&b&0\\0&0&0&0&\dots&0&0&c\\0&0&0&0&\dots&0&0&0\end{pmatrix},$$

$$A(d,b,c,\gamma)=\begin{pmatrix}-d\gamma&c&0&0&\dots&0&0&d\\0&0&b&0&\dots&0&0&0\\0&0&0&b&\dots&0&0&0\\0&0&0&0&\dots&0&0&0\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots\\0&0&0&0&\dots&0&b&0\\-c\gamma&0&0&0&\dots&0&0&c\\-d\gamma^2&c\gamma&0&0&\dots&0&0&d\gamma\end{pmatrix},\quad A_j=A(d_j,b_j,c_j,\gamma_j)$$

and the following equalities and inequalities hold (the first five express the fact that the sum of the residua is equal to zero, the sixth is the right hand-side of (21)):

$$1+\sum_{j=1}^{p}b_j=0,\quad -1+\sum_{j=1}^{p}d_j\gamma_j=0,\quad \sum_{j=1}^{p}d_j(\gamma_j)^2=0,$$
$$\sum_{j=1}^{p}d_j=0,\quad \sum_{j=1}^{p}c_j\gamma_j=0,\quad \sum_{j=1}^{p}d_j/(a_j-a_{p+1})=0,$$
$$c_j\neq 0,\quad b_j\neq 0,\quad d_j\neq 0,\quad \gamma_j\neq 0,\quad \gamma_j\neq\gamma_k\quad \text{for}\, j\neq k.$$

The reader will easily check that such a choice is possible (e.g. one can choose $\gamma_j^2=1+1/(a_j-a_{p+1})$ etc.) and for almost all such choices

1) the residua $A_1,\dots,A_p$ and the operators $M_1,\dots,M_p$ are conjugate to Jordan blocks $n\times n$;

2) the operator M_{p+1} is conjugate to a Jordan block $n\times n$;

3) the monodromy group of the system is irreducible (hint – none of the sums $\sum_{k=s}^{n}\sum_{j=1}^{p+1}(\beta_{kj}+\varphi_{kj})$ is a non-negative integer because $\varphi_{1,p+1}$ is greater than the other $\varphi_{k,p+1}$ $(k>1)$, i.e. (16) does not hold; these sums correspond to the invariant subspaces of M_{p+1}).

Hence, the set $\Sigma_{0,n}^*$ is non-empty for any $n\geq 2$; it contains the intersection of $\tilde{\Sigma}$ with the hypersurface $\{h=0\}$ in $(\mathrm{GL}(n,\mathbb{C}))^p$. Hence, $\operatorname{codim}_{\Sigma}\Sigma_0=1$ and $\operatorname{codim}_{(\mathrm{GL}(n,\mathbb{C}))^p}\Sigma_0=2p(n-1)$. The theorem is proved. □

6.3 The cases $n = 4, 5, 6$

In this subsection we consider the strata which are or possibly are bad for $n = 4, 5, 6$. These are the cases for which Theorem 6.6 does not say whether the main bad strata are indeed the most significant (i.e. the only bad ones of highest dimension). For reducibility patterns $\left(\begin{smallmatrix} a & N \\ 0 & Q \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} Q & N \\ 0 & a \end{smallmatrix}\right)$, Q being $(n-1) \times (n-1)$, $n = 4, 5, 6$, the only bad strata are the main ones (see Corollary 5.4). An easy computation shows that any reducibility pattern of the kind $\operatorname{diag}(P_1, \dots, P_s)$, $s \leq 6$, P_i being its big blocks (i.e. a direct sum), has a higher codimension in $(\mathrm{GL}(n, \mathbb{C}))^p$, $n = 4, 5, 6$, than the one of the main bad strata. We consider the rest of the possible reducibility patterns.

(A) $n = 4$, *reducibility pattern* $\left(\begin{smallmatrix} P & Q \\ 0 & R \end{smallmatrix}\right)$ (*all blocks are* 2×2). A stratum is bad only if the P- and Q-blocks of each operator M_j have the same eigenvalues, of proportional multiplicities (see Corollary 5.4). For each j there are two possibilities:

1) M_j is conjugate to a Jordan matrix of two Jordan blocks 2×2 (if to one of the two eigenvalues there correspond two blocks 1×1, then the Riemann–Hilbert problem has a positive solution, see Corollary 5.6);

2) M_j has one eigenvalue of multiplicity 4.

If we are in the first case at least for one j and if $\varphi_1, \dots, \varphi_4$ are the diagonal entries of the matrix A_j, see Theorem 2.7, then for the system to be Fuchsian at a_j it is necessary and sufficient to have $\varphi_1 \geq \varphi_3$, $\varphi_2 \geq \varphi_4$ (assuming that these are the couples of integers φ_j corresponding to the same Jordan blocks), see condition (7); we assume that (6) is fulfilled; the given regular system is not equivalent to a Fuchsian one if one cannot find any $\varphi_1, \dots, \varphi_4$ such that $\varphi_1 = \varphi_3$, $\varphi_2 = \varphi_4$ and (6) holds. The codimension of such systems is very high (see [22]) and for them the codimension $2p(n-1)$ from Theorem 6.5 is not attained.

If we are in case 2) for all $j = 1, \dots p+1$, then we must have $\varphi_1 \geq \varphi_2 \geq \varphi_3 \geq \varphi_4$. Equality (8) implies that we must have equalities everywhere. The codimension of the stratum equals

$$\kappa_1 + \kappa_2, \quad \kappa_1 = 4(p-1), \quad \kappa_2 = 3(p+1) - 1$$

(κ_1 comes from the reducibility pattern, κ_2 – from the equalities between the eigenvalues; we leave the details for the reader). We have $\kappa_1 + \kappa_2 = 7p - 2 > 6p - 1$ ($6p - 1$ is the codimension of the main bad strata in $(\mathrm{GL}(4, \mathbb{C}))^p$).

We let the reader prove oneself that in the case of other reducibility patterns the codimension of the potentially bad strata is at least $\kappa_1' + \kappa_2'$, $\kappa_2' = 2(p+1) - 1$ (Corollary 5.6), $\kappa_1' \geq 5(p-1)$ coming from the reducibility pattern.

E.g., if the reducibility pattern is

$$\begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}$$

(i.e. one big and three small blocks) and the distribution of the eigenvalues among the small blocks is a, b, (a, b) (for all j, with two Jordan blocks 2×2), then the codimension of the stratum is exactly $\kappa_1' + \kappa_2' = 7p - 4$. Denote by $\varphi_{1,j}$ and $\varphi_{3,j}$ the numbers φ corresponding to the eigenvalues a and by $\varphi_{2,j}$ and $\varphi_{4,j}$ the ones corresponding to b. Then (if the system is Fuchsian) one has

$$\sum_{j=1}^{p+1}(\beta_{1,j} + \varphi_{1,j}) \leq 0, \quad \sum_{k=1}^{2}\sum_{j=1}^{p+1}(\beta_{k,j} + \varphi_{k,j}) \leq 0,$$
$$\varphi_{1,j} \geq \varphi_{3,j}, \quad \varphi_{2,j} \geq \varphi_{4,j}, \quad \sum_{k=1}^{4}\sum_{j=1}^{p+1}(\beta_{k,j} + \varphi_{k,j}) = 0.$$

Hence, $\varphi_{1,j} = \varphi_{3,j}, \varphi_{2,j} = \varphi_{4,j}$ and $\sum_{j=1}^{p+1}(\beta_{k,j} + \varphi_{k,j}) = 0$ for $k = 1, 2, 3, 4$. This means that there exists an equivalence (2) with a constant matrix W which brings the given Fuchsian system to a block upper-triangular form (the same as the reducibility pattern), see Theorem 5.1.2 from [12].

However, there exist positions of the poles for which these equalities do not hold for the block 2×2, i.e. one cannot find a Fuchsian system 2×2 with monodromy group defined by the small blocks 2×2 of the monodromy matrices M_j and whose eigenvalues satisfy the above equalities. For fixed poles such monodromy groups form a codimension 1 subset in the set of all monodromy groups with the given reducibility pattern and given Jordan normal forms and distribution of the eigenvalues of the monodromy matrices.

Hence, their codimension in the space of monodromy groups is $7p - 3$. The codimension of the subset of bad groups from the main bad strata equals $6p$. Hence, for $p = 3$ these codimensions coincide.

(B) $n = 5$, *reducibility pattern* $\begin{pmatrix} P & Q \\ 0 & R \end{pmatrix}$, P *is* 2×2, R *is* 3×3. (If P is 3×3 and R is 2×2 the reasoning is the same.) For the potentially bad strata the eigenvalues of P and R must be the same for each M_j. Hence, either two of the eigenvalues of R are equal to one of the eigenvalues of P and the third is equal to the other eigenvalue of P, or the operator M_j has only one eigenvalue. In the first case, using Corollary 5.4, we see that the stratum is good. In the second one the codimension is minimal (we do not discuss the question whether the stratum is bad or not) if M_j is conjugate to a Jordan block 5×5, for all $j = 1, \ldots, p + 1$. The codimension of this stratum is $6(p - 1) + 4(p + 1) - 1 = 10p - 3 > 8p - 1$. (The codimension of the main bad strata equals $8p - 1$.)

For other reducibility patterns we leave the estimation of the codimension for the reader. In all cases the one of the main bad strata is smaller.

(C) $n = 6$, *reducibility pattern* $\begin{pmatrix} P & Q \\ 0 & R \end{pmatrix}$.

(C1) P *and* Q *are* 3×3. A stratum is bad only if for each M_j there exist at least 3 equalities between its eigenvalues, see Corollary 5.4. Hence, its codimension in $(\mathrm{GL}(6, \mathbb{C}))^p$ is at least $9(p - 1) + 3(p + 1) - 1 = 12p - 7 \geq 10p - 1$ for $p \geq 3$

($10p - 1$ is the codimension of the main bad strata). There is equality of these codimensions only for $p = 3$.

(C2) *P is 2×2 and Q is 4×4 or vice versa.* A stratum is bad only if for each M_j there exist at least 4 equalities between its eigenvalues, see Corollary 5.4. The codimension is at least $8(p-1) + 4(p+1) - 1 = 12p - 5 > 10p - 1$.

It is easy to prove for other reducibility patterns that the codimensions of the possible bad strata are greater than the one of the main bad ones. The list being too long, we prefer to restrict ourselves to the above examples.

6.4 The case $p = 2$ – some examples

In the present subsection we give two examples showing how Theorems 6.5 and 6.6 can be extended (with the necessary modifications) to the case $p = 2$. The first example is constructed by incomplete analogy with part 4^o of the proof of Theorem 6.5.

Example 6.8. The Fuchsian system with three poles

$$\dot{X} = \left(\begin{pmatrix} 1 & 0 & 0 \\ a & 0 & 0 \\ b & c & -1 \end{pmatrix} \Big/ t + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Big/ (t-1) + \begin{pmatrix} -1 & -1 & 0 \\ -a & 0 & -1 \\ -b & -c & 1 \end{pmatrix} \Big/ (t-2) \right) X \tag{22}$$

and its monodromy group have the following two properties:

Property A. All three monodromy operators M_1, M_2, M_3 are conjugate to Jordan blocks of size 3 with eigenvalue 1 if $a + b - c = 0$ and $a + c + 1 = 0$.

The property can be checked directly.

Property B. The system is not equivalent to a Fuchsian system with the same poles and with all three matrices-residua conjugate to nilpotent Jordan blocks of size 3.

Indeed, the matrix $W(t)$ of the equivalence (2) should be of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ h/t & 0 & 1 \end{pmatrix},$$

$h \in \mathbb{C}$ (modulo equivalences (2) with constant matrices W). However, the fact that the entries in position $(1, 3)$ of the three matrices-residua are 0 implies that such an equivalence does not exist.

System (22) depends on 1 parameter (there are two equations satisfied by a, b, c). One can conjugate the system by matrices from $\mathrm{SL}(3, \mathbb{C})$ which gives a 9-dimensional family of systems which realize monodromy groups having Properties A and B.

At the same time the dimension of the variety of triples of matrices (M_1, M_2, M_3), $M_1M_2M_3 = I$, having Property A equals 10. This means that for $p = 2$, $n = 4$ Theorem 6.5 is true. Indeed, system (22) realizes the block P of a monodromy group from the set Σ_0.

Similar examples can be constructed for all $n \geq 3$, but not for $n = 2$ (the matrices-residua will be lower-triangular and the monodromy group of the system will be reducible). The details from this example are left for the reader.

Example 6.9. This example is inspired by Example 5.3.1 of A. A. Bolibrukh from [1], p. 105. For $p = 2, n = 4$, consider a monodromy group defining the reducibility pattern $\left(\begin{smallmatrix} P & Q \\ 0 & R \end{smallmatrix}\right)$ (the blocks P and R are 2×2). Suppose that the operators M_1, M_2 have each in their Jordan normal form two Jordan blocks of size 2, with different eigenvalues – b, b^{-1} – where each of the blocks P and R of M_1, M_2 has eigenvalues b, b^{-1}. Suppose that M_3 is conjugate to a Jordan block of size 4 with eigenvalue -1.

Suppose that b is close to 1. The exponents $\beta_{k,j}$ are such that $\exp(2\pi i \beta_{k,j}) = b^{\pm 1}$, so one can set $\beta_{k,j} = \beta$ for $j = 1, 2$, $k = 1, 3$ and $\beta_{k,j} = -\beta$ for $j = 1, 2$, $k = 2, 4$, where β is close to 0. One has $\beta_{k,3} = 1/2$, $k = 1, \ldots, 4$.

One has

$$0 \geq \sum_{k=1}^{2} \sum_{j=1}^{3} \varphi_{k,j} + \beta_{1,3} + \beta_{2,3} = \sum_{k=1}^{2} \sum_{j=1}^{3} \varphi_{k,j} + 1 \in \mathbb{Z},$$

see (16); we use here the fact that $\sum_{k=1}^{2} \sum_{j=1}^{2} \beta_{k,j} = 0$. Hence, one has also

$$0 \leq \sum_{k=3}^{4} \sum_{j=1}^{3} \varphi_{k,j} + 1 \in \mathbb{Z},$$

see (8). It follows from condition (7) that one must have $\varphi_{1,3} = \varphi_{2,3} = \varphi_{3,3} = \varphi_{4,3}$ and $\varphi_{1,j} = \varphi_{3,j}, \varphi_{2,j} = \varphi_{4,j}$ for $j = 1, 2$. Hence, $\sum_{k=3}^{4} \sum_{j=1}^{3} \varphi_{k,j} + 1 = 2 \sum_{j=1}^{3} \varphi_{k,j} + 1 = 0$ which is impossible because $\varphi_{k,j} \in \mathbb{Z}$.

The codimension in $(\mathrm{GL}(4, \mathbb{C}))^2$ of the set of bad groups from this example equals $4 + 2 + 2 + 3 = 11$ (4 comes from the reducibility pattern; there are respectively 2, 2 and 3 equalities between eigenvalues of M_1, M_2 and M_3). The codimension of Σ_0 equals 12.

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Monodromy of Cherednik–Kohno–Veselov connections

Vladimir P. Leksin

Department of Algebra and Geometry
Kolomna State Pedagogical Institute
Kolomna, Zelenaya, 30, 140411, Russia
email: `lexine@mccme.ru`

In memory of Andrei Bolibrukh

Abstract. A special class of flat logarithmic connections ∇_R on $\mathbb{C}^n$ associated with finite complex vector configurations $R \subset \mathbb{C}^n$ generating $\mathbb{C}^n$ is considered. The connection ∇_R acts on the trivial holomorphic bundle with fiber $\mathbb{C}^n$ and it has logarithmic poles on hyperplanes that are orthogonal to vectors of R with respect to the standard Hermitian form of $\mathbb{C}^n$. We prove that Veselov's $\vee$-conditions for a complex vector configuration R are equivalent to the Frobenius integrability of the connection ∇_R. If R is a root system with finite complex reflection group $W(R)$, then R satisfies Veselov's conditions and ∇_R is an integrable connection. In the case of some root systems R, we describe the monodromy representation of the generalized braid group $B_n(R)$ defined by the associated logarithmic connection $\overline{\nabla}_R$ on the quotient space $\mathbb{C}^n / W(R)$. These representations are deformations of the standard representations of the corresponding complex reflection groups. They are generalizations of the Burau representations for some complex root systems which were earlier defined by Squier and Givental only for real root systems.

Contents

Introduction

To any configuration R of non-zero vectors generating the complex vector space $\mathbb{C}^n$, one may attach a logarithmic connection on the trivial holomorphic bundle on $\mathbb{C}^n$ with

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fiber $\mathbb{C}^n$

$$\nabla_R(\underline{h}) = d - \sum_{\alpha\in R} h_\alpha \frac{d(z,\ \alpha)}{(z,\ \alpha)}\alpha^*\alpha, \tag{$*$}$$

where $(u, v) = \sum_{i=1}^n u_i\bar{v}_i$ is the standard Hermitian form in $\mathbb{C}^n$, operator coefficients $\alpha^*\alpha$ of the connection form are constant operators $(\alpha^*\alpha)(v) = (v, \alpha)\alpha$ of rank one on $\mathbb{C}^n$ and $\underline{h} = (h_\alpha)_{\alpha\in R}$ are arbitrary complex parameters. The superscript " $*$ " denotes the complex conjugation of vector coordinates followed by the transposition (α has one line and n columns) or, in other words, α^* is the dual vector to α with respect to the standard Hermitian form.

If the connection ∇_R is integrable in the Frobenius sense (in other words, a flat connection) then we call it a *Cherednik–Kohno–Veselov connection* or more simply a *R-connection.*

Similar connections are considered for real vector configurations by Veselov in [20] for finding special solutions of the generalized WDVV equations and by Cherednik, Kohno in [3], [4], [11] when they consider some reductions of the KZ connections and their generalizations. The relation between R-connections and KZ connections is based on the equalities $\alpha^*\alpha = \frac{(\alpha,\alpha)}{2}(1 - s_\alpha)$ and $\alpha^*\alpha = \frac{(\alpha,\alpha)}{p}\big(\sum_{j=1}^{p-1}\det(s_\alpha^{-j})s_\alpha^j\big)$. Here s_α is a complex reflection $s_\alpha(v) = v - \frac{2(v,\alpha)}{(\alpha,\alpha)}\alpha$ of order 2 or $s_\alpha(v) = v - (1 - \zeta_{p(\alpha)})\frac{(v,\alpha)}{(\alpha,\alpha)}\alpha$, $\zeta_{p(\alpha)} = \exp\frac{2\pi i}{p(\alpha)}$, is a complex reflection of order $p(\alpha)$ respectively.

We prove that when $h_\alpha > 0$ for each α, $\nabla_R(\underline{h})$ is integrable iff a certain vector configuration $\tilde{R}$ satisfies certain conditions due to Veselov. The root systems of finite complex reflection groups (in particular, the Coxeter root systems) are the main examples of vector configurations satisfying Veselov's conditions. Independently of this result, we prove the integrability of the R-connections for root systems R of complex reflection groups $W(R)$ and complex parameters $(h_\alpha)_{\alpha\in R}$ satisfying the invariance condition

$$h_{w(\alpha)} = h_\alpha \quad \text{for all } \alpha\in R,\ w\in W(R).$$

The connection $\nabla_R(\underline{h})$ then descends to the quotient space $\mathbb{C}^n/W(R)$, which is isomorphic to $\mathbb{C}^n$ by the Shephard–Todd generalization of the Chevalley theorem. The quotient connection has logarithmic singularities on the discriminant divisor $D = \big(\bigcup_{\alpha\in R}\{z\in\mathbb{C}^n \mid (z,\alpha) = 0\}\big)/W(R)$. It gives rise to a representation of the generalized braid group $B_n(R) = \pi_1(\mathbb{C}^n\setminus D)$ (here $\pi_1(X)$ is the fundamental group of a space X). This representation is called *the monodromy representation of the generalized braid group $B_n(R)$* of the R-connection $\nabla_R(\underline{h})$. For a well-generated complex reflection group $W(R)$ (that is, generated by n reflections $s_{\alpha_1},\dots,s_{\alpha_n}$ for some basis

$$\{\alpha_1,\dots,\alpha_n \mid \alpha_j\in R,\ j = 1,\dots,n\}$$

in $\mathbb{C}^n$) we consider the generalized Cartan matrix

$$K(R) = \left(\frac{(1-\zeta_{p(\alpha_m)})(\alpha_k,\alpha_m)}{(\alpha_m,\alpha_m)}\right)_{k,m=1,\dots,n}$$

and its deformation $K_q(R) = K^- + qK^+$, where K^- is the lower-triangular part of K with diagonal part diag $(1, \ldots, 1)$ and K^+ is the upper-triangular part of $K(R)$ with diagonal part diag$(-\zeta_{p(\alpha_1)}, \ldots, -\zeta_{p(\alpha_n)})$. If the Cohen–Dynkin diagram (see [9], [10]) of a complex root system R is a tree, then using K_q we define the generalized Burau representation $B_n(R) \to \mathrm{GL}_n(\mathbb{C}[q, q^{-1}])$ of the generalized braid group $B_n(R)$ in automorphisms of the free module of rank n over the ring of Laurent polynomials $\mathbb{C}[q, q^{-1}]$. For real Coxeter root systems similar representations were defined by Givental (ADE Coxeter types) (see [8]) and by Squier (arbitrary Coxeter types) (see [17]). Our main result is the description of monodromy representations of the generalized braid groups $B_n(R)$ as generalized Burau representations corresponding to the root system R under certain restrictions on R.

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1 Generalized Veselov conditions and integrable R-connections

Let $R = \{\alpha_1, \ldots, \alpha_N\} \subset \mathbb{C}^n$ be a finite configuration of non-zero vectors of the complex vector space $\mathbb{C}^n$ generating $\mathbb{C}^n$. Let us fix the standard Hermitian form $(u, v) = \sum_{i=1}^n u_i \bar{v}_i$ on $\mathbb{C}^n$ and identify $\mathbb{C}^n$ with dual space $(\mathbb{C}^n)^*$ with respect to the standard Hermitian form (the complex structure on $(\mathbb{C}^n)^*$ is conjugate to the complex structure on $\mathbb{C}^n$). Denote by v^* the dual vector to vector v with respect to the standard Hermitian form such that if $v = (v_1, \ldots, v_n)$ then $v^* = (\bar{v}_1, \ldots, \bar{v}_n)$. We suppose that all vectors $\alpha \in R$ have unit length $|\alpha| = \sqrt{(\alpha, \alpha)} = 1$.

Let $L \subset \mathbb{C}^n$ be a two-dimensional subspace of $\mathbb{C}^n$. Define the linear operator P_L by the equality $P_L = \sum_{\alpha \in R \cap L} \alpha^* \alpha$. It maps $\mathbb{C}^n$ to the subspace L and, in particular, L is an invariant subspace with respect to P_L.

Now we describe the Veselov condition ($\vee$-conditions) for a vector configuration R (where vectors are non-zero, but do not necessarily have unit length).

Definition 1. A vector configuration R satisfies the generalized Veselov condition, if for any two-dimensional subspace $L \subset \mathbb{C}^n$ the vectors $\alpha \in R \cap L$ are eigenvectors of the operator P_L. In this case we will call R a Veselov system.

One easily sees that Veselov's condition is equivalent the following conditions:

1) If the linear span $\mathrm{Span}(L \cap R) \subset \mathbb{C}^n$ has complex dimension 1, then Veselov's condition is obviously satisfied.

2) If this dimension is 2, and there exist two non-colinear vectors $\alpha, \beta \in L \cap R$ such that for any vector $\gamma \in L \cap R$ we have $\gamma = \lambda\alpha$ or $\gamma = \lambda\beta$ with some $\lambda \in \mathbb{C}^\times$ then Veselov's condition is equivalent to the orthogonality of these vectors $(\alpha, \beta) = 0$.

3) If the intersection $R \cap L$ contains more than two pairwise non-colinear vectors from R, then the restriction of the linear operator $P_L = \sum_{\alpha \in R \cap L} \alpha^* \alpha$ to the subspace

L is proportional to the identity. We denote by $\lambda(L)$ the complex number such that $P_L|_L = \lambda(L)\,\mathrm{id}_L$.

Denote by $\Omega_R(\underline{h}) = \sum_{\alpha\in R} h_\alpha \frac{d(z,\alpha)}{(z,\alpha)}\alpha^*\alpha$ the 1-form of the R-connection with complex parameters h_α, $\alpha \in R$.

A generalization of the observation in [20] about the relation between the Veselov condition on a configuration R and the integrability of the R-connection is proved in the following proposition.

Proposition 2. *Suppose that all parameters h_α are positive real numbers. Then the connection $\nabla_R(\underline{h}) = d - \Omega_R(\underline{h})$ is integrable if and only if the vector configuration $\tilde{R} = \{\sqrt{h_\alpha}\alpha \mid \alpha \in R\}$ satisfies the Veselov condition.*

Proof. We use the following well-known fact (see [19], [14]): the closed logarithmic differential 1-form $\Omega = \sum_{\alpha\in\tilde{R}} t_\alpha \frac{d(\alpha,z)}{(\alpha,z)}$ with constant operator coefficients t_α, $\alpha \in R$ satisfies the Frobenius condition $\Omega\wedge\Omega = 0$ if and only if for any two-dimensional subspace $L \subset \mathbb{C}^n$ the following equalities are fulfilled

$$\left[t_\alpha, \sum_{\beta\in\tilde{R}\cap L} t_\beta\right] = 0 \quad \text{for all } \alpha \in R\cap L. \tag{1}$$

Let us prove that the integrability of $\nabla_R(\underline{h})$ (where $t_\alpha = \alpha^*\alpha$ for any $\alpha \in \tilde{R}$) implies that $\tilde{R}$ satisfies the Veselov condition. For this, we will prove that it satisfies the set of conditions 1), 2), 3) above. Condition 1) is obviously satisfied. Let us check condition 2). Assume that $L \subset \mathbb{C}^2$ contains non-colinear $\alpha, \beta \in \tilde{R}$, such that any $\gamma \in L\cap\tilde{R}$ is colinear to α or β. Then for some $x, y \in \mathbb{R}_+^\times$, we have $\sum_{\gamma\in\tilde{R}\cap L} t_\gamma = x t_\alpha + y t_\beta$. We know that $\left[t_\alpha, \sum_{\gamma\in\tilde{R}\cap L} t_\gamma\right] = 0$, hence $y[t_\alpha, t_\beta] = 0$. Now for any $v \in \mathbb{C}^n$, $[t_\alpha, t_\beta](v) = (\alpha,\beta)\big((\beta,v)\alpha - (\alpha,v)\beta\big)$. Since α and β are non-colinear, this operator is zero iff $(\alpha,\beta) = 0$. Hence condition 2) is satisfied.

Let us now show condition 3). Assume that $L \subset \mathbb{C}^2$ contains 3 pairwise non-colinear vectors α, β and γ from $\tilde{R}$. We know that $\left[t_\lambda, \sum_{\delta\in\tilde{R}\cap L} t_\delta\right] = 0$ for $\lambda = \alpha, \beta$ or γ. The operators t_λ and $\sum_{\delta\in\tilde{R}\cap L} t_\delta$ both restrict to endomorphisms of L; $\sum_{\delta\in\tilde{R}\cap L} t_\delta \colon L \to L$ coincides with P_L and $t_\lambda \colon L \to L$ coincides with $\lambda^*\lambda$. Hence $[P_L, \lambda^*\lambda] = 0$ for $\lambda = \alpha, \beta$ or γ. Since $\dim(L) = 2$, the set of operators in $\mathrm{End}(L)$ commuting with $\lambda^*\lambda$ is $\mathrm{Span}(\mathrm{id}_L, \lambda^*\lambda) = \{a\,\mathrm{id}_L + b\lambda^*\lambda \mid a, b \in \mathbb{C}\}$. As α, β and γ are non-colinear, they cannot be pairwise orthogonal. Assume for example that $(\alpha,\beta) \neq 0$. Then $(\mathrm{id}_L, \alpha^*\alpha, \beta^*\beta)$ is a free family of $\mathrm{End}(L)$. It follows that $\mathrm{Span}(\mathrm{id}_L, \alpha^*\alpha) \cap \mathrm{Span}(\mathrm{id}_L, \beta^*\beta) = \mathbb{C}\,\mathrm{id}_L$. Hence $P_L \in \mathbb{C}\,\mathrm{id}_L$, so 3) is satisfied.

Let us now prove that if $\tilde{R}$ satisfies the Veselov condition, then $\nabla_R(\underline{h})$ is flat. Let us assume that $\tilde{R}$ satisfies the Veselov condition. Let $L \subset \mathbb{C}^n$ be a two-dimensional vector subspace and let us show that for all $\alpha \in \tilde{R}\cap L$, $\left[t_\alpha, \sum_{\gamma\in\tilde{R}\cap L} t_\gamma\right] = 0$. If the vectors from $\tilde{R}$ is L are all colinear, then $\sum_{\gamma\in\tilde{R}\cap L} t_\gamma$ is proportional to t_α and this equality holds. If $L\cap\tilde{R}$ contains non-colinear vectors, and there exist $\alpha_0, \beta_0 \in L\cap\tilde{R}$ such that any vector of $L\cap\tilde{R}$ is colinear to α_0 or β_0, then we know from condition 2) that $(\alpha_0, \beta_0) = 0$. Then $\sum_{\gamma\in\tilde{R}\cap L} t_\gamma = x t_{\alpha_0} + y t_{\beta_0}$, for $x, y > 0$; now $(\alpha_0, \beta_0) = 0$

implies $[t_{\alpha_0}, t_{\beta_0}] = 0$; therefore $\big[t_{\alpha_0}, \sum_{\gamma\in\tilde R\cap L} t_\gamma\big] = \big[t_{\beta_0}, \sum_{\gamma\in\tilde R\cap L} t_\gamma\big] = 0$. Now since any t_α, $\alpha \in \tilde R \cap L$, is proportional to t_{α_0} or t_{β_0}, we get $\big[t_\alpha, \sum_{\gamma\in\tilde R\cap L} t_\gamma\big] = 0$ for all $\alpha \in \tilde R \cap L$. Finally, assume that $L \cap \tilde R$ contains more than 3 non-colinear vectors. Let $\alpha \in \tilde R \cap L$. Both operators t_α and $\sum_{\gamma\in\tilde R\cap L} t_\gamma$ preserve the decomposition $\mathbb{C}^n = L \oplus L^\perp$. Moreover, their restrictions to $L^\perp$ are both zero, so the restriction of $\big[t_\alpha, \sum_{\gamma\in\tilde R\cap L} t_\gamma\big]$ to $L^\perp$ is zero; and condition 3) implies that the restriction of $\sum_{\gamma\in\tilde R\cap L} t_\gamma$ to L is proportional to the identity, hence the restriction of $\big[t_\alpha, \sum_{\gamma\in\tilde R\cap L} t_\gamma\big]$ to L also vanishes. Hence $\big[t_\alpha, \sum_{\gamma\in\tilde R\cap L} t_\gamma\big] = 0$. Hence $\nabla_R(\underline{h})$ is flat. □

The basic examples of vector configurations satisfying the Veselov conditions are the root systems of finite complex reflection groups (see [9], [10]). Let us recall their definition. Let $R \subset \mathbb{C}^n$ be a finite configuration of unit vectors in $\mathbb{C}^n$ generating $\mathbb{C}^n$ and $p\colon R \to \mathbb{N} \setminus \{0, 1\}$ be a function with integer values $\geqslant 2$.

For $\alpha \in R$, set $s_{\alpha,p(\alpha)}(v) = v - (1 - \zeta_{p(\alpha)})(v, \alpha)\alpha$, where $\zeta_{p(\alpha)} = \exp \frac{2\pi i}{p(\alpha)}$. Then $\mathcal{S} = \{s_{\alpha,p(\alpha)}, \alpha \in R\}$ is a set of unitary complex reflections. We suppose that: (a) the set $\mathcal{S}$ generates a finite subgroup $W(R)$ in the unitary group $U(n, \mathbb{C})$, (b) for each $\alpha \in R$, $s_{\alpha,p(\alpha)}(R) = R$ and (c) $p(s_{\beta,p(\beta)}(\alpha)) = p(\alpha)$ for all $\alpha, \beta \in R$. Then a pair (R, p) is called a *complex pre-root system* of $W(R)$. A pre-root system (R, p) is called a *complex root system* of the complex reflection group $W(R)$ if addition $\lambda\alpha \in R$ if and only if $\lambda\alpha \in W(R)\alpha$ for all $\alpha \in R$, $\lambda \in \mathbb{C}^\times$.

Theorem 3. *Let (R, p) be a complex root system of the complex reflection group $W(R)$ and $\underline{h} = (h_\alpha)_{\alpha\in R}$ be an arbitrary $W(R)$-invariant map $R \to \mathbb{C}$. Then the R-connection $\nabla_R(\underline{h})$ is integrable. In particular, if $h_\alpha > 0$ for any $\alpha \in R$, then the vector configuration $\tilde R = \{\sqrt{h_\alpha}\alpha \mid \alpha \in R\}$ satisfies the Veselov condition.*

Proof. Recall that
$$\nabla_R(\underline{h}) = d - \sum_{\alpha\in R} h_\alpha \frac{d(z, \alpha)}{(z, \alpha)} \alpha^*\alpha.$$

Set $r_\alpha = \alpha^*\alpha$ for $\alpha \in \tilde R$. Then $t_\alpha = h_\alpha r_\alpha$, and we will show that for any two-dimensional subspace $L \subset \mathbb{C}^n$ and any $\alpha \in R \cap L$, we have $\big[r_\alpha, \sum_{\beta\in R\cap L} h_\beta r_\beta\big] = 0$. In what follows, we write s_α instead of $s_{\alpha,p(\alpha)}$. For any integer i, we have $s_\alpha^i = 1 - (1 - \zeta_{p(\alpha)}^i) r_\alpha$, $\det(s_\alpha^i) = \zeta_{p(\alpha)}^i$. Then
$$r_\alpha = \frac{1}{p(\alpha)} \sum_{j=0}^{p(\alpha)-1} \det(s_\alpha^{-j}) s_\alpha^j. \tag{2}$$

We have $s_\alpha r_\beta = r_{s_\alpha(\beta)} s_\alpha$. Then
$$s_\alpha\big(\textstyle\sum_{\beta\in R\cap L} h_\beta r_\beta\big) = \big(\sum_{\beta\in R\cap L} h_\beta r_{s_\alpha(\beta)}\big) s_\alpha = \big(\sum_{\beta\in R\cap L} h_\beta r_\beta\big) s_\alpha.$$
The last equality follows from the facts that $s_\alpha\colon R \to R$ is bijective, and $s_\alpha(L) = L$, so that $s_\alpha\colon R \cap L \to R \cap L$ is bijective, and from the $W(R)$-invariance of $\alpha \mapsto h_\alpha$.

Then (2) implies that $\left[r_\alpha, \sum_{\beta\in R\cap L} h_\beta r_\beta\right] = 0$, as wanted. Then Proposition 1 implies the second part of the theorem. □

Remark 4. Let us define a non-degenerate Hermitian form G_R on $\mathbb{C}^n$ associated with a generating vector configuration R by

$$G_R(u, v) = \sum_{\alpha\in R} (u, \alpha)(\alpha, v)$$

and endomorphisms of $\mathbb{C}^n$

$$(\alpha^\vee\alpha)(u) = G_R(u, \alpha)\alpha \quad \text{for all } \alpha \in R.$$

By direct calculation we can verify that the R-connection with coefficients $t_\alpha = \alpha^\vee\alpha$ is integrable if and only if the function on $\mathbb{C}^n$

$$F_R(z_1, \dots, z_n) = \sum_{\alpha\in R} |(z, \alpha)|^2 \log |(z, \alpha)|^2$$

satisfies the generalized WDVV equations (see [20])

$$[\hat{F}_i, \hat{F}_j] = 0, \quad 1 \leqslant i < j \leqslant n.$$

Here the matrices $\hat{F}_i(z)$, $i = 1, \dots, n$, are defined by $\hat{F}_i(z) = [G_R]^{-1} F_i(z)$ where $F_i(z) = \left(\frac{\partial^3 F_R}{\partial z_i \partial z_k \partial \bar{z}_l}\right)_{k,l=1,\dots,n}$, and $[G_R]$ is the matrix of the Hermitian form G_R in the canonical basis of $\mathbb{C}^n$.

2 Monodromy of *R*-connections for complex root systems

In 1954, G. Shephard and J. Todd (see [16]) completely classified irreducible finite complex reflection groups. They showed that there are exist the three infinite families of the irreducible complex reflection groups (cyclic groups C_n, symmetric groups Σ_n and imprimitive groups $G(m, k, n)$, where m, k, n are positive integers such that k divides m) and 34 exceptional groups $G_4, \dots, G_{37}$. The irreducible finite real Coxeter reflection groups are naturally contained in the Shephard–Todd classification.

In 1976, A. Cohen [5] presented a root system corresponding to every complex reflection group of dimension greater than two but the four-dimensional reflection group G_{31}, which only has an extended root system with five generating roots). In [10], for most irreducible two-dimensional reflection groups (12 out of 19), a root system was obtained. All complex reflection groups having a root system are well-defined complex reflection groups, that is, they have n generators in their presentations (the root system generates $\mathbb{C}^n$). The analogues of the sets of positive and simple roots (in the case of real root systems) are called, in the case of complex root systems, the set of primary roots and a basis of this set. Using bases of primary roots, the Cohen–Dynkin diagrams were defined in [5], [9], [10] for complex root systems.

We will study only well-defined complex reflection groups having n simple roots in a root system. We also demand that the Cohen–Dynkin diagram of the root system is union of a trees (that is, in each diagram there is no cycle). All the Coxeter reflection groups and the following complex reflection groups in the Shephard–Todd classification: $G(m,1,n)$, $G(m,m,n)$, $G_4, \ldots, G_{10}$, G_{14}, G_{16}, G_{17}, G_{18}, G_{20}, G_{21}, G_{25}, G_{26}, G_{32} have such diagrams. In this case all complex reflection groups have a Coxeter-like presentation

$$\langle s_1, \ldots, s_n \mid s_i^{p_i} = 1,\ s_i s_j s_i \ldots = s_j s_i s_j \ldots,\ 1 \leqslant i < j \leqslant n \rangle \tag{3}$$

with m_{ij} factors on each side of the second set of relations (see [2]).

Now, for any complex root system (R, p) we consider the primitive root set $R_0 \subset R$ consisting from vectors of the R of the length one and we select one vector that is an eigenvector for each reflection $s_{\alpha,p(\alpha)}(v) = v - (1 - \zeta_{p(\alpha)})(v,\alpha)\alpha$ with eigenvalue $\zeta_{p(\alpha)}$ different from 1.

The generalized pure braid group $P_n(R)$ is defined as the fundamental group $P_n(R) := \pi_1\big(\mathbb{C}^n \setminus \bigcup_{\alpha\in R}\{(v,\alpha)=0\}\big)$ of the complement in $\mathbb{C}^n$ of the union of reflection hyperplanes $\mathcal{H} = \bigcup_{\alpha\in R}\{v|(v,\alpha)=0\}$. The complex reflection group $W(R)$ acts freely on the complement $Y_n(R) = \mathbb{C}^n \setminus \mathcal{H}$. The generalized braid group is defined as the fundamental group $B_n(R) = \pi_1(X_n(R))$ of the quotient space $X_n(R) = Y_n(R)/W(R)$. The homotopy exact sequence of the covering $Y_n(R) \to X_n(R)$ gives rise to the short exact sequence of groups

$$1 \longrightarrow P_n(R) \longrightarrow B_n(R) \longrightarrow W(R) \longrightarrow 1.$$

Under our assumptions on the complex root system R, we have a presentation

$$B_n(R) = \langle \sigma_1, \ldots, \sigma_n \mid \underbrace{\sigma_i\sigma_j\sigma_i \ldots}_{m_{ij}\text{ factors}} = \underbrace{\sigma_j\sigma_i\sigma_j \ldots}_{m_{ij}\text{ factors}},\ 1 \leqslant i < j \leqslant n \rangle. \tag{4}$$

Let $\alpha_1, \ldots, \alpha_n$ be a set of "simple" roots (recall, with unit lengths) of the complex root system R, that is, $\alpha_1, \ldots, \alpha_n$ is a basis of the linear span of R_0, and therefore of the linear span of R. We consider the generalized Cartan matrix $K(R)$ of the root system R. It is defined by

$$K(R) = \big((1 - \zeta_{p(\alpha_m)})(\alpha_k, \alpha_m)\big)_{\{k,m=1,\ldots,n\}}. \tag{5}$$

Let use decompose $K(R) = K^- + K^+$, where K^- is lower-triangular with units on the diagonal and K^+ is upper-triangular with diagonal $-\zeta_{p(\alpha_1)}, \ldots, -\zeta_{p(\alpha_n)}$. The Givental-Squier deformation $K_q(R)$ of $K(R)$ is given by (see [8], [17])

$$K_q(R) = K^- + qK^+, \tag{6}$$

where q is formal (or complex) parameter.

Let us consider the free module $\mathbb{C}^n \otimes \mathbb{C}[q,q^{-1}] = \mathbb{C}[q,q^{-1}]^{\oplus n}$ of rank n over the ring of Laurent polynomials $\mathbb{C}[q,q^{-1}]$. The operation of complex conjugation * extends to $\mathbb{C}[q,q^{-1}]$ by $q^* = q^{-1}$. Let $\mathbb{C}(q)$ be the quotient field of $\mathbb{C}[q,\ q^{-1}]$.

Let us write $v \in \mathbb{C}(q)^{\oplus n}$ as a column vector; the canonical basis of $\mathbb{C}(q)^{\oplus n}$ is $\{e_1 = \alpha_1, \dots, e_n = \alpha_n\}$. Let us define a sesquilinear form by $\langle u, v\rangle = u^t K_q \bar{v}$ where v^t is the transpose of v.

We define n linear transformations of $\mathbb{C}(q)^{\oplus n}$ by formulas

$$\rho_B^R(\sigma_j)(v) = v - \langle v, e_j\rangle e_j. \tag{7}$$

These are invertible transformations of $\mathbb{C}(q)^{\oplus n}$. In fact, we have

$$\rho_B^R(\sigma_j)(v + q^{-1}\zeta_{p(\alpha_j)}^{-1}(v^t K_q \bar{e}_j)e_j) = v.$$

It follows from this equality that the linear transformations $\rho_B(\sigma_j)$ satisfy the equations

$$(\rho_B^R(\sigma_j))^2 - (q\zeta_{p(\alpha_j)} + 1)\rho_B^R(\sigma_j) + q\zeta_{p(\alpha_j)} = 0. \tag{8}$$

We see also that each matrix $(\rho_B^R(\sigma_j)$, $j = 1, \dots, n$ satisfies the equation

$$(\rho_B^R(\sigma_j) - 1)(\rho_B^R(\sigma_j) - q\zeta_{p(\alpha_j)}) = 0. \tag{8'}$$

Remark 5. If $\zeta_{p(\alpha_1)} = \zeta_{p(\alpha_2)} = \cdots = \zeta_{p(\alpha_n)} = \zeta$, then the linear transformations $\rho_B^R(\sigma_j)$ are also unitary operators with respect to the generalized Hermitian form $\langle u, v\rangle = u^t K_q \bar{v}$ on $\mathbb{C}(q)^{\oplus n}$ defined with matrix $K_q(R)$; in this case $K_q(R)$ satisfies $K_q^* = (-q^{-1}\zeta^{-1})K_q$.

The generalized Cartan matrix K_q is a nondegenerate matrix, since the highest degree coefficient in $\det K_q(R)$ (i.e., the coefficient of q^n) is

$$(-1)^n \zeta_{p(\alpha_1)} \dots \zeta_{p(\alpha_n)} \neq 0.$$

For complex root systems, we have the following generalization of

(a) Kohno's theorem ([11], for real Coxeter root system of the type A);

(b) Givental's theorem ([8], for real Coxeter root systems of types A, D, or E);

(c) Squier's theorem ([17], for arbitrary real Coxeter root systems).

Theorem 6. *If the Cohen–Dynkin diagram of complex root system R is a tree, then the transformations $\rho_B^R(\sigma_i)$, $i = 1, \dots, n$, define a representation of the generalized braid group $B_n(R)$ in the group* $\mathrm{Aut}(\mathbb{C}(q)^{\oplus n}) = \mathrm{GL}_n(\mathbb{C}(q))$ *of automorphisms of the vector space $\mathbb{C}(q)^{\oplus n}$ over the field $\mathbb{C}(q)$.*

Proof. We must verify that the $\rho_B^R(\sigma_i)$ satisfy the relations of the generalized braid group $B_n(R)$ in presentation (4). We will follow Squier's plan [17] for the proof of such relations. The first step consists in Squier's reduction to the case of operators acting on a two-dimensional subspace of $\mathbb{C}(q)^{\oplus n}$. The second step is based on the Coxeter and Hughes–Morris inductive methods for the verification of relations of well-generated two-dimensional finite complex reflection groups (see [6] and also [10]) and the homogeneity of formulas with respect to the parameter q.

For given i, j satisfying $1 \leqslant i < j \leqslant n$ we denote by V_{ij} the subspace in $\mathbb{C}(q)^{\oplus n}$ spanned by α_i and α_j. We consider the orthogonal submodule $V_{ij}^{\perp}$ to the submodule V_{ij} with respect to the sesquilinear form defined by $K_q(R)$. We have $V_{ij} \cap V_{ij}^{\perp} = 0$. Indeed, if $v = v_i\alpha_i + v_j\alpha_j \in V_{ij}$ and $v \in V_{ij}^{\perp}$, then $v^t K_q(R)\alpha_i^* = 0$ and $v^t K_q(R)\alpha_j^* = 0$ which leads to system of linear equations

$$\begin{aligned}(1 - q\zeta_{p(\alpha_i)})(\alpha_i, \alpha_i)v_i + (1 - \zeta_{p(\alpha_i)})(\alpha_j, \alpha_i)v_j = 0,\\ (1 - \zeta_{p(\alpha_j)})q(\alpha_i, \alpha_j)v_i + (1 - q\zeta_{p(\alpha_j)})(\alpha_j, \alpha_j)v_j = 0.\end{aligned}$$

Recall that $(\alpha_i, \alpha_i) = 1$, $i = 1, \ldots, n$. Since the determinant of the coefficient matrix is

$$\zeta_{p(\alpha_i)}\zeta_{p(\alpha_j)}q^2 + q\Big((\zeta_{p(\alpha_i)} + \zeta_{p(\alpha_j)})(|(\alpha_i, \alpha_j)|^2 - 1) - (\zeta_{p(\alpha_i)}\zeta_{p(\alpha_j)} + 1)|(\alpha_i, \alpha_j)|^2 - |(\alpha_i, \alpha_j)|^2\Big) + 1 \neq 0$$

in $\mathbb{C}(q)$, the only solution is $v_i = v_j = 0$, so $v = 0$, as required. The subspaces V_{ij} and $V_{ij}^{\perp}$ are invariant with respect to operators $\rho_B^R(\sigma_i)$ and $\rho_B^R(\sigma_j)$. On the subspace $V_{ij}^{\perp}$ operators $\rho_B^R(\sigma_i)$ and $\rho_B^R(\sigma_j)$ are identity operators. Let us denote by A and B the matrices of operators $\rho_B^R(\sigma_i)$ and $\rho_B^R(\sigma_j)$ in V_{ij} with respect to the basis e_i, e_j. We have

$$A = \begin{pmatrix} q\zeta_{p(\alpha_i)} & 0 \\ -k_{ji} & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & -qk_{ij} \\ 0 & q\zeta_{p(\alpha_j)} \end{pmatrix},$$

where $k_{ji} = (1 - \zeta_{p(\alpha_i)})(\alpha_j, \alpha_i)$ and $k_{ij} = (1 - \zeta_{p(\alpha_j)})(\alpha_i, \alpha_j)$

For brevity, we use Coxeter's notation from [6] $r = \zeta_{p(\alpha_i)}$, $s = \zeta_{p(\alpha_j)}$, $u = -k_{ji}$ and $v = -k_{ij}$. We obtain

$$A = \begin{pmatrix} qr & 0 \\ u & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & qv \\ 0 & qs \end{pmatrix}.$$

For some matrices A and B as in [6] we obtain following formulas

$$A(BA)^{n-1} = \begin{pmatrix} qra_n & qrvc_{n-1} \\ uc_n & b_n \end{pmatrix}, \quad B(AB)^{n-1} = \begin{pmatrix} a_n & qvc_n \\ qsuc_{n-1} & qsb_n \end{pmatrix},$$
$$(AB)^n = \begin{pmatrix} qra_n & q^2rvc_n \\ uc_n & b_{n+1} \end{pmatrix}, \qquad (BA)^n = \begin{pmatrix} a_{n+1} & qvc_n \\ qsuc_n & qsb_n \end{pmatrix},$$

where $c_0 = 0$, $a_1 = b_1 = c_1 = 1$. Multiplying on the left or on the right by A or B we obtain the inductive system of equations in a_n, b_n and c_n

$$\begin{aligned} a_{n+1} &= qra_n + quvc_n,\\ b_{n+1} &= qsb_n + quvc_n,\\ c_n &= a_n + qsc_{n-1} = b_n + qrc_{n-1}.\end{aligned}$$

These equations are easily reduced to

$$\begin{aligned} a_n &= c_n - qsc_{n-1}, \\ b_n &= c_n - qrc_{n-1}, \\ c_{n+1} &= q(uv + r + s)c_n - q^2 rsc_{n-1}. \end{aligned}$$

Solving the last linear inductive equation of the order two, with initial conditions $c_0 = 0,\ c_1 = 1$, we obtain

$$c_n = q^{n-1}\left(\frac{1}{d}\left(\left(\frac{\tilde{c}_2 + d}{2}\right)^n + \left(\frac{\tilde{c}_2 - d}{2}\right)^n\right)\right),$$

where $\tilde{c}_2 = uv + r + s$, $d = \sqrt{\tilde{c}_2{}^2 - 4rs}$. If in the relation

$$\underbrace{\sigma_i\sigma_j\sigma_i\ldots}_{m_{ij}\text{ factors}} = \underbrace{\sigma_j\sigma_i\sigma_j\ldots}_{m_{ij}\text{ factors}}$$

from the presentation of the generalized braid group, the number of factors $m_{ij} = 2n$ is even, then we observe that a sufficient condition for $(AB)^n = (BA)^n$ is

$$\frac{\tilde{c}_2 + d}{\tilde{c}_2 - d} = e^{\frac{2\pi\sqrt{-1}}{n}},$$

that gives $c_n = 0$ for each n. This sufficient condition for the generators of the reflection group is achieved choosing $e'_i = e_i$, $e'_j = (1 - \zeta_{p(\alpha_j)})(\alpha_i, \alpha_j)e_j$ when corresponding values of k_{ij} and k_{ji} are equal to the values u and v found by Coxeter in [6] or by Hughes and Morris in [10].

If $m_{ij} = 2n - 1$ is odd, then we must have $p(\alpha_i) = p(\alpha_j)$ (that is, $r = s$) and the sufficient condition has the form $\tilde{c}_n - q\sqrt{rs}\tilde{c}_{n-1} = 0$, whence $a_n = b_n = 0$ and the corresponding relation for A and B is fulfilled. The Coxeter values for u and v guarantee the last sufficient condition $\tilde{c}_n - \sqrt{rs}\tilde{c}_{n-1} = 0$. The proof is complete. □

We will call representations ρ_B^R of the generalized braid groups $B_n(R)$ the *generalized Burau representations*.

Let us note that when the parameter q in the Burau representation ρ_B^R is equal to one, then the Burau representation factors through the corresponding complex reflection group $W(R)$ and it coincides with the standard action of $W(R)$ on $\mathbb{C}^n$.

If R is an irreducible complex root system, we consider the R-connection

$$\nabla_R(\underline{h}) = d - \sum_{\alpha\in R} h_\alpha \frac{d(z,\alpha)}{(z,\alpha)}\alpha^* \otimes \alpha \tag{9}$$

in the trivial vector bundle

$$\pi : \left(\mathbb{C}^n \setminus \textstyle\bigcup_{\alpha\in R}\{z \in \mathbb{C}^n \mid (z,\alpha) = 0\}\right) \times \mathbb{C}^n \longrightarrow \mathbb{C}^n \setminus \textstyle\bigcup_{\alpha\in R}\{z \in \mathbb{C}^n \mid (z,\alpha) = 0\}.$$

A complex reflection group $W(R)$ associated with a complex root system R naturally acts on the base $Y_n = \mathbb{C}^n \setminus \bigcup_{\alpha\in R}\{z \in \mathbb{C}^n \mid (z, \alpha) = 0\}$ and on the fiber $F_n = \mathbb{C}^n$ of the trivial bundle π. We suppose that the complex parameters $\{h_\alpha,\ \alpha \in R\}$ are $W(R)$-invariant (that is, $h_{w((\alpha))}) = h_\alpha$ for all $w \in W(R)$).

By Theorem 2, the R-connection $\nabla_R(\underline{h})$ is a flat connection. This flat connection defines a representation $\rho_R(\underline{h})$ of the generalized pure braid group $P_n(R) = \pi_1(Y_n)$

$$\rho_R(\underline{h})\colon P_n(R) \longrightarrow \mathrm{Aut}(\mathbb{C}^n) = \mathrm{GL}(n, \mathbb{C}).$$

The flat connection $\nabla_R(\underline{h})$ is invariant with respect to the action of $W(R)$, therefore, it defines a flat quotient-connection $\overline{\nabla}_R(\underline{h})$ on the quotient-bundle

$$\begin{aligned}\bar{\pi}\colon \big(\big(\mathbb{C}^n \setminus {\textstyle\bigcup_{\alpha\in R}}\{(z, \alpha) = 0\}\big) \times \mathbb{C}^n\big)/W(R)& \\ \longrightarrow \big(\mathbb{C}^n \setminus {\textstyle\bigcup_{\alpha\in R}}\{(z, \alpha) = 0\}\big)/W(R).&\end{aligned}$$

The monodromy representation of the connection $\overline{\nabla}_R(\underline{h})$ extends the representation $\rho_R(\underline{h})$ to a representation $\bar{\rho}_R(\underline{h})$ of the generalized braid group (see [3], [2], [15])

$$\bar{\rho}_R(\underline{h})\colon B_n(R) \longrightarrow \mathrm{Aut}(\mathbb{C}^n) = \mathrm{GL}(n, \mathbb{C}).$$

For small values of parameters $h_\alpha, \alpha \in R$ or their generic values, the representation $\bar{\rho}_R(\underline{h})$ factors via the cyclotomic Hecke algebra $H_n(q_k,\ k = 1, \dots, n)$, as was proved by Broué, Malle and Rouquier (see [2], [15]). The cyclotomic Hecke algebra is defined as the following quotient algebra of the group algebra of the generalized braid group:

$$\begin{aligned}&H_n(q_k,\ k = 1, \dots, n)\\ &\quad= \mathbb{C}[B_n(R)]\big/\big\{(\sigma_k - 1)(\sigma_k - q_k\zeta_{p(\alpha_k)}) \textstyle\prod_{j=2}^{p(\alpha_k)-1}(\sigma_k - \zeta^j{}_{p(\alpha_k)}) = 0,\\ &\qquad\qquad q_k = \exp\Big(\frac{2\pi i h_{\alpha_k}}{p(\alpha_k)}\Big),\ k = 1, \dots, n\big\},\end{aligned}$$

where $\{\alpha_1, \dots, \alpha_n \mid \alpha_k \in R_0,\ (\alpha_k, \alpha_k) = 1,\ k = 1, \dots, n\}$ is the set of simple roots of the root system R. Thus, we have a commutative diagram

$$\begin{array}{ccccc} B_n(R) & \xrightarrow{\bar{\rho}_R(\underline{h})} & \mathrm{Aut}(\mathbb{C}^n) = \mathrm{GL}(n, \mathbb{C}) & \xrightarrow{\subset} & \mathrm{End}(\mathbb{C}^n) \\ {\scriptstyle j}\big\downarrow & & & & \big\uparrow{\scriptstyle \widehat{\bar{\rho}}_R(\underline{h})} \\ \mathbb{C}[B_n(R)] & \xrightarrow[\phi]{} & H_n(q_k,\ k = 1, \dots, n) & = & H_n(q_k,\ k = 1, \dots, n) \end{array}$$

where $\widehat{\bar{\rho}}_R(\underline{h})$ is a representation of the cyclotomic Hecke algebra $H_n(q_k,\ k = 1, \dots, n)$.

If all parameters h_{α_k} are equal to 0, then the cyclotomic Hecke algebra $H_n(q_k = 1,\ k = 1, \dots, n)$ is isomorphic to the group algebra $\mathbb{C}[W(R)]$ of the complex reflection group $W(R)$ and the representations $\rho_R(\underline{0})$, $\widehat{\bar{\rho}}_R(\underline{0})$ coincide with the standard action of $W(R)$ and $\mathbb{C}[W(R)]$ on $\mathbb{C}^n$.

Note that it follows from equation (8′) that the generating matrices $\rho_B^R(\sigma_j)$ of the generalized Burau representation satisfy relations of the cyclotomic Hecke algebra.

Since the product of the first two factors of the corresponding relation already give zero (see also the proof of theorem below), the generalized Burau representation factors through the cyclotomic Hecke algebra.

If the cyclotomic Hecke algebra $H_n(q_k,\ k = 1, \dots, n)$ and the corresponding group algebra $\mathbb{C}[W(R)]$ of the complex reflection group are isomorphic for generic values for the parameters, then the following theorem is true. This theorem describes the monodromy representation of the connection (9). Such an isomorphism holds in the following cases: real finite Coxeter groups (see [4]), infinite series of finite complex reflection groups (see [2], [15]), all two-dimensional complex reflection groups (see [7]) and some other exceptional groups (see references in [2], [15]).

Theorem 7. *Let $W(R)$ be an irreducible well-generated complex reflection group associated with a irreducible complex root system R, such that the Cohen–Dynkin diagram of R is a tree. Let $\nabla_R(\underline{h})$ be the R-connection* (9) *associated with R and $h_\alpha = h$ for all $\alpha \in R$, $h \in \mathbb{C}$. We suppose also that the set of parameters $q_k = \exp\left(\frac{2\pi i h}{p(\alpha_k)}\right)$ is generic for the Hecke algebra $H_n(q_k,\ k = 1, \dots, n)$. Then the monodromy representation $\rho_R(h)$ of the R-connection $\nabla_R(\underline{h})$ is equivalent to the generalized Burau representation ρ_B^R with value of the parameter $q = \exp(2\pi i h)$.*

Proof. It follows from equations (8) and (8′) that the generalized Burau representation ρ_B^R with parameter $q = \exp(2\pi i h)$ factors via the cyclotomic Hecke algebra $H_n(q_k,\ k = 1, \dots, n)$ with pointed parameters q_k because $q = q_k^{p(\alpha_k)}$, $k = 1, \dots, n$ and $(\rho_B^R(\sigma_k) - 1)(\rho_B^R(\sigma_k) - q\zeta_{p(\alpha_k)}) = 0$, where $\zeta_{p(\alpha_k)} = \exp \frac{2\pi i}{p(\alpha_k)}$. The monodromy representation of the $\nabla_R(\underline{h})$ and the generalized Burau representation with $h = 0$ coincide with the standard representation of complex reflection group $W(R)$. If there exists an isomorphism between the cyclotomic Hecke algebra $H_n(q_k,\ k = 1, \dots, n)$ and the group algebra $\mathbb{C}[W(R)]$, then the theorem follows from the following commutative diagram

$$
\begin{array}{ccccccc}
B_n(R) & \xrightarrow{\bar{\rho}_R(\underline{h})} & \mathrm{GL}(n, \mathbb{C}) & \xrightarrow{\subset} & \mathrm{End}(\mathbb{C}^n) & = = = & \mathrm{Mat}(n, \mathbb{C}) \\
{\scriptstyle j}\big\downarrow & & & & {\scriptstyle \hat{\bar{\rho}}_R(\underline{h})}\big\uparrow{\scriptstyle \hat{\rho}_B^R} & & \big\uparrow{\scriptstyle \hat{\rho}} \\
\mathbb{C}(B_n(R)) & \xrightarrow[\phi]{} & H_n(q) & = = = & H_n(q) & \xrightarrow[\simeq]{} & \mathbb{C}[W(R)],
\end{array}
$$

where $\hat{\rho}$ is the standard representation of the group algebra $\mathbb{C}[W(R)]$ in the algebra $\mathrm{End}(\mathbb{C}^n) = \mathrm{Mat}(n, \mathbb{C}^n)$. □

As mentioned above, if all parameters $p(\alpha_j)$, $j = 1, \dots n$ of some complex reflection group are equal to some positive integer p then the form $\langle u, v\rangle = u^t K_q \bar{v}$ is a generalized Hermitian form and the monodromy representations of the corresponding R-connection will be equivalent to generalized unitary representations. This is the case for all Coxeter groups and also for some imprimitive complex reflection groups

$G(m, 1, n)$, and seven exceptional irreducible complex reflection groups (five two-dimensional ones, G_4, G_5, G_8, G_{16} and G_{20}, and two multi-dimensional ones, G_{25} and G_{32}).

For real Coxeter root systems the description of the monodromy representation of R-connections similar to Theorem 3 was obtained by V. Toledano Laredo in [18].

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Invitation to Galois theory

Hiroshi Umemura

Graduate School of Mathematics
Nagoya University, Japan
email: `umemura@math.nagoya-u.ac.jp`

Abstract. This note is introductory and prepared for non-specialists. We explain by using examples the basic ideas of a general differential Galois theory of ordinary differential field extensions.

Contents

1 History

No one can doubt that the Galois theory of algebraic field extensions is one of the most fundamental elements of number theory. Construction of an analogous theory in analysis is a historical problem going back to the 19th century. So it would be useful to have a look at the history of Galois theory to understand the present status.

We owe the Galois theory of algebraic equations to two mathematicians who died prematurely, N. Abel (1802–1829) and E. Galois (1811–1832). They founded the Galois theory of algebraic equations and applied it to quintic equations. They showed that the solution of the general quintic equation is not reducible to extraction of radicals and the four arithmetic operations $\pm$, $\times$, $\div$. Thus they put an end to the historical problem of finding a formula expressing rationally roots of a quintic equation using radicals. However this was not an end but a starting point of modern number theory.

S. Lie (1842–1899) noticed the importance of the rich ideas of Galois and Abel and he had a plan of constructing a similar theory for differential equations. It was a hard task to realize the plan. One of the main reasons for the difficulties is that the theory is *infinite dimensional* and he did not have even finite dimensional tools such as the theory of finite dimensional Lie groups and Lie algebras. So he had to begin by founding finite dimensional theories. E. Picard (1865–1941) was the first who realized a part of Lie's idea. He presented a Galois theory of linear ordinary differential equations in 1896 called today Picard–Vessiot theory. This is a nice theory but it is *finite dimensional.* Then a mysterious mathematician J. Drach followed Picard. In his thesis [4] published in 1898, Drach proposed a Galois theory of non-linear ordinary differential equations. His theory is *infinite dimensional.* So this is the first trial of *infinite dimensional* differential Galois theory or general differential Galois theory. As E. Vessiot pointed out, the thesis of Drach contains incomplete definitions and gaps in proofs. So Vessiot immediately started to review Drach's work. He received Le Grand Prix of Academy of Paris in 1902 for his work on infinite dimensional differential Galois theory. He devoted his life to infinite dimensional differential Galois theory.

While Vessiot pursued infinite dimensional differential Galois theory, an unusual event in mathematics, a debate, took place in Comptes Rendus in 1902. The dispute between P. Painlevé and R. Liouville on the irreducibility of the first Painlevé equation continued for several months. We can summarize the opinion of Painlevé at the final stage of the dispute as follows.

(1) The first Painlevé equation is irreducible with respect to the so far known functions.

(2) Drach's infinite dimensional differential Galois theory is still incomplete. However it should be put in a clear form before long so that it could be widely accepted.

(3) The irreducibility question is best settled in the framework of Drach's theory.

Painlevé was too optimistic in this opinion. In fact, 100 years later, today, we have infinite dimensional differential Galois theories such as Malgrange's theory [7] and ours [10] but we do not know whether they are popular or not. The irreducibility of the first Painlevé equation was proved in 1980's but the proof does not depend on infinite dimensional differential Galois theory (cf. [6], [11] and Section 8).

Research in differential Galois theory of *infinite* dimension was active in the early years of the last century. Vessiot published his last papers in 1947. Infinite dimensional differential Galois theory was abandoned for several decades until J.-F. Pommaret published his book [8] in 1983.

Ritt (1893–1955) planted a branch of the tree on the other side of the Atlantic. Kolchin (1916–1993), a successor of Ritt, made two major contributions. (i) Using the language of algebraic geometry of A. Weil, he accomplished *finite dimensional* differential Galois theory. (ii) Together with Ritt, he founded differential algebra (cf. [5]).

In the 1960s, Jacobson, Sweedler, Bourbaki et al. constructed Galois theories of inseparable field extensions. The idea of these theories is to replace finite group by finite group scheme or by Hopf algebra. So far as dimension is concerned, these theories are of dimension 0.

We published in 1996 a differential Galois theory of infinite dimension [10]. We were inspired by an idea of Vessiot [14] and developed it in the language of schemes. Our theory is a Galois theory of differential field extensions.

Malgrange too proposed a Galois theory of infinite dimension [7]. He also started from another idea of Vessiot. His Galois theory is a Galois theory of foliations on a manifold. You also find a Galois theory of P. Cassidy and M. Singer [3] in this volume. See Section 7 for a relation between their theory and ours.

All the rings that we consider are commutative and contain 1 and the field $\mathbb{Q}$ of rational numbers.

2 Galois theory of algebraic equations

Let us recall the principal ideas of Galois theory of algebraic equations by a simple example. Given a cubic equation

$$x^3 + 6x^2 - 8 = 0, \tag{1}$$

we can easily check that the polynomial

$$f(x) := x^3 + 6x^2 - 8 \in \mathbb{Q}[x]$$

is irreducible over $\mathbb{Q}$ and the equation (1) has 3 distinct real roots. As far as we look at one particular solution $x = \alpha$, we cannot see any symmetry. On the other hand, if we introduce the set

$$S := \{(x_1, x_2, x_3) \mid \text{the } x_i\text{'s are distinct roots of equation (1) for } 1 \le i \le 3\},$$

then the symmetric group S_3 of degree 3 operates naturally on the set S. Namely for $g \in S$ and $x = (x_1, x_2, x_3) \in S$,

$$g \cdot x = (x_{g(1)}, x_{g(2)}, x_{g(3)}) \in S.$$

Moreover the operation (S_3, S) is a principal homogeneous space, by which we mean that if we take an element $x \in S$, then the map

$$S_3 \to S, \quad g \mapsto g \cdot x$$

is a bijection.

Let us recall that the discriminant D of a general cubic equation

$$a_0x^3 + a_1x^2 + a_2x + a_3 = 0, \quad a_0 \neq 0$$

is equal to

$$D := (x_1 - x_2)^2(x_1 - x_3)^2(x_2 - x_3)^2$$

by definition. On the other hand, we can express the discriminant D in terms of the coefficients a_i, $1 \leq i \leq 3$, namely

$$D = a_1^2 a_2^2 + 18a_0a_1a_2a_3 - 4a_0a_2^3 - 4a_1^3a_3 - 27a_0^2a_3^2. \tag{2}$$

If we calculate the discriminant of the algebraic equation (1) by (2), then

$$(x_1 - x_2)^2(x_1 - x_3)^2(x_2 - x_3)^2 = 2^6 3^6.$$

So

$$(x_1 - x_2)(x_1 - x_3)(x_2 - x_3) = \pm 2^3 3^3 \tag{3}$$

for $x = (x_1, x_2, x_3) \in S$ according to the order of the 3 roots. We take an element $x \in S$ and fix it once and for all. An algebraic relation among roots with coefficients in $\mathbb{Q}$ is called a constraint so that (3) is a constraint. Classically the Galois group G of the algebraic equation (1) is a subgroup of the symmetric group S_3 consisting of all the elements that leave all the constraints invariant. In particular we have

$$G \subset A_3 = \{g \in S_3 \mid (x_{g(1)} - x_{g(2)})(x_{g(1)} - x_{g(3)})(x_{g(2)} - x_{g(3)}) \\ = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)\}.$$

In this case it is easy to show that the Galois group G coincides with the alternating group A_3.

Here is a summary of our observations.

(1) S_3 symmetry of the cubic equation becomes apparent when we introduce the set S.

(2) The operation (S_3, S) is a principal homogeneous space.

(3) The Galois group is a subgroup of S_3 consisting all the elements of S_3 that leave every constraint invariant.

Above we followed the heuristic argument. The most elegant way of defining the Galois group, due to Dedekind, is

$$G := \mathrm{Aut}(\mathbb{Q}(x_1, x_2, x_3)/\mathbb{Q}),$$

where the right-hand side is the automorphism group of the field extension $\mathbb{Q}(x_1, x_2, x_3)$ of the field $\mathbb{Q}$ of rational numbers.

3 Galois theory of linear differential equations, Picard–Vessiot theory

Let us consider a linear ordinary differential equation

$$y'' = xy \tag{4}$$

of second order. Here x is the independent variable so that

$$y' = dy/dx, \quad y'' = d^2y/dx^2, \dots.$$

If we look at a single solution of (4), we can not see any symmetry as for the algebraic equation (1). We introduce the set

$$S := \left\{ Y(x) = \begin{bmatrix} y_1(x) & y_2(x) \\ y_1(x)' & y_2(x)' \end{bmatrix} \;\middle|\; \begin{array}{l} y_i(x) \text{ is a solution of (4) holomorphic} \\ \text{in a neighborhood of } x = 0 \in \mathbb{C} \\ \text{for } i = 1, 2 \text{ such that } \det Y(x) \neq 0. \end{array} \right\}$$

Hence we have

$$Y'(x) = \begin{bmatrix} 0 & 1 \\ x & 0 \end{bmatrix} Y(x). \tag{5}$$

The general linear group $GL_2(\mathbb{C})$ operates on the set S in an evident way. For $g \in GL_2(\mathbb{C})$, $Y(x) \in S$, the result of the operation is the product $Y(x)g$ of the 2×2-matrices. Moreover $(GL_2(\mathbb{C}), S)$ is a principal homogeneous space.. Now it follows from (5) that

$$(\det Y(x))' = \operatorname{tr} \begin{bmatrix} 0 & 1 \\ x & 0 \end{bmatrix} \det Y(x)$$

so that

$$(\det Y(x))' = 0. \tag{6}$$

Namely for $Y(x) \in S$, $\det Y(x)$ is a constant. Now we take an element $Y(x) \in S$ and fix it. The Galois group G of the linear ordinary differential equation (4) is the subgroup of $GL_2(\mathbb{C})$ consisting of all the elements that leave all the constraints invariant. We mean by a constraint an algebraic relation among the entries of the matrix $Y(x)$ with coefficients in $\mathbb{C}(x)$. So in particular $\det Y(x) = c \in \mathbb{C}$ is a constraint. Hence we can conclude that

$$G \subset \{g \in GL_2(\mathbb{C}) \mid \det Y(x)g = \det Y(x)\} = SL_2(\mathbb{C}).$$

It is not evident but we can show in this case that we have in fact

$$G = SL_2(\mathbb{C})$$

(cf. p. 29, [9]). Here is a summary of our observations.

(1) $GL_2(\mathbb{C})$ symmetry of the linear differential equation becomes apparent when we introduce the set S.

(2) The operation $(GL_2(\mathbb{C}), S)$ is a principal homogeneous space.

(3) The Galois group is a subgroup of GL_2 consisting all the elements of GL_2 that leave every constraint invariant.

We can also define the Galois group as an automorphism group of field extension. Let $Y(x)$ be an element of S. We denote by L the field generated by $y_1(x)$, $y_2(x)$, $y_1'(x)$, $y_2'(x)$ over $K := \mathbb{C}(x)$, namely

$$L = K(y_1(x), y_2(x), y_1'(x), y_2'(x)) = \mathbb{C}(x, y_1(x), y_2(x), y_1'(x), y_2'(x)).$$

Hence both K and L are closed under the derivation, in other words, they are differential fields (cf. Section 6). L/K is a differential field extension. Now we have

$G = \mathrm{Aut}(L/K)$, the right-hand side being the group of field automorphisms $f : L \to L$ commuting with the derivation and leaving every element of the subfield K fixed. The differential field L is differentially generated over K by a basis of solutions of the linear differential equation (4). Such an extension is called a Picard–Vessiot extension.

4 General Galois theory of non-linear ordinary differential equations

Let us consider a non-linear ordinary differential equation.

$$y'' = F(x, y, y'), \tag{7}$$

where $F(x, y, y')$ is a polynomial in x, y, y' with coefficients in $\mathbb{C}$. A particular case of this type of equation is the first Painlevé equation $y'' = 6y^2 + x$. The question is how to reveal the hidden symmetry of the non-linear differential equation (7). After a reflection one is convinced that introducing the set of all the solutions of (7) is not a good idea. Instead, we choose and fix a point $x_0 \in \mathbb{C}$ in general position and we set

$$\begin{aligned} S(x_0) :=\{(y(w_1, w_2; x), y_x(w_1, w_2; x)) \mid\ & y(w_1, w_2; x) \text{ is a solution holomorphic at} \\ & x = x_0 \text{ of (7) containing two parameters } w_1, \\ & w_2 \text{ such that } D(y, y_x)/D(w_1, w_2) \neq 0.\} \end{aligned}$$

Here

$$\frac{D(f, g)}{D(w_1, w_2)} = \begin{vmatrix} f_{w_1} & g_{w_1} \\ f_{w_2} & g_{w_2} \end{vmatrix}$$

is the Jacobian. In other words, we consider the local solution $y(w_1, w_2; x)$ holomorphic at $x = x_0$ containing initial conditions w_1, w_2. We also need the set

$$\begin{aligned} \Gamma_2 := \{\Phi(w) = (\varphi_1(w_1, w_2), \varphi_2(w_1, w_2)) \mid (w_1, w_2) & \mapsto \Phi(w_1, w_2) \\ & \text{is a coordinate transformation}\}. \end{aligned}$$

To be more precise, $w = (w_1, w_2) \mapsto \Phi(w)$ is an analytic isomorphism between two open subsets U, V of $\mathbb{C}^2$. So we better write

$$\Phi = \Phi_{U,V} : U \xrightarrow{\sim} V.$$

Hence if

$$\Phi_{U,V}, \Phi_{W,X} \in \Gamma_2$$

such that $V \subset W$, then we can compose $\Phi_{U,V}$ and $\Phi_{W,X}$ to get

$$\Phi_{W,X} \circ \Phi_{U,V} \in \Gamma_2.$$

Γ_2 is an example of a Lie pseudo-group. As we can not necessarily compose two elements of Γ_2, it is not a group but it is almost a group. The Lie pseudo-group Γ_2

almost operates, pseudo-operates, on the set S. Namely for

$$\Phi \in \Gamma_2, \quad \boldsymbol{y} := (y(w_1, w_2; x), y_x(w_1, w_2; x)) \in S,$$

we define

$$\Phi \cdot \boldsymbol{y} = (y(\varphi_1(w_1, w_2); x), y_x(\varphi_2(w_1, w_2); x)). \tag{8}$$

In the definition of the operation (8), we should be careful of the domains of definition of Φ and $y(w_1, w_2; x)$. If $\Phi \in \Gamma_2$ defines an isomorphism $U \xrightarrow{\sim} V$ of two open subsets of $\mathbb{C}^2$ and if $y(w_1, w_2; x)$ and $y_x(w_1, w_2; x)$ are regular on an open subset of $\mathbb{C}^3$ containing $V \times x_0$, then

$$\Phi \cdot (y(w_1, w_2; x), y_x(w_1, w_2; x)) \in S(x_0).$$

We say that the pseudo-group Γ_2 pseudo-operates on the set $S(x_0)$. Moreover, $(\Gamma_2, S(x_0))$ is almost a principal homogeneous space in the following sense. Let

$$\boldsymbol{y}_i = (y_i(w_1, w_2; x), y_{i,x}(w_1, w_2; x)) \in S(x_0)$$

for $i = 1, 2$, then there exists locally a unique element $g \in \Gamma_2$ such that $g \cdot \boldsymbol{y}_1 = \boldsymbol{y}_2$.

We have seen so far that the pseudo-operation $(\Gamma_2, S(x_0))$ reveals the symmetry of the non-linear equation (7). This suggests that we might define the Galois group of (7) as a Lie pseudo-subgroup of Γ_2 consisting all the transformations that leave all the constraints invariant. Let us look at this idea closely. We must first clarify a constraint. Let us take an element

$$\boldsymbol{y} = (y(w_1, w_2; x), y_x(w_1, w_2; x)) \in S(x_0).$$

Definition 4.1. A constraint with respect to an element $\boldsymbol{y} \in S(x_0)$ is an algebraic relation between the derivatives

$$\frac{\partial^{l+m+n} y(w_1, w_2; x)}{\partial x^l \partial w_1^m \partial w_2^n}, \quad l, m, n = 0, 1, 2, \ldots,$$

with coefficients in $\mathbb{C}[[w_1, w_2]][x]$.

This definition looks natural but if we adopt this it, the definition of a constraint depends heavily on the choice of the element

$$\boldsymbol{y} = (y(w_1, w_2; x), y_x(w_1, w_2; x)) \in S(x_0).$$

In the previous nice examples, algebraic equations and linear differential equations, an element of the set S is related with another by a rational relation so that they define the same algebraic constraints. For example for an algebraic equation, let $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in S$, then they differ only by the ordering of the roots so that $\{x_1, x_2, x_3\} = \{y_1, y_2, y_3\}$. Hence the field extension $\mathbb{Q}(x_1, x_2, x_3)$ coincides with $\mathbb{Q}(y_1, y_2, y_3)$. Also for the linear differential equation two elements of S define the same differential field extension. In the non-linear case (7), however, let

$$\boldsymbol{y}_i = (y_i(w_1, w_2; x), y_{i,x}(w_1, w_2; x)) \in S(x_0)$$

for $i = 1, 2$ be two elements of $S(x_0)$. As we explained above, they are related through a transformation $\Phi \in \Gamma_2$: $\Phi \cdot \boldsymbol{y}_1 = \boldsymbol{y}_2$. What makes the non-linear case different is the fact that the transformation Φ is transcendental so that we can not expect that the differential field extension

$$\mathbb{C}((w_1, w_2))\left(x, \frac{\partial^{l+m+n}\boldsymbol{y}_1(w_1, w_2; x)}{\partial x^l \partial w_1^m \partial w_2^n}\right)_{l,m,n,=0,1,2,\ldots} \Big/ \mathbb{C}((w_1, w_2))(x)$$

be isomorphic to

$$\mathbb{C}((w_1, w_2))\left(x, \frac{\partial^{l+m+n}\boldsymbol{y}_2(w_1, w_2; x)}{\partial x^l \partial w_1^m \partial w_2^n}\right)_{l,m,n,=0,1,2,\ldots} \Big/ \mathbb{C}((w_1, w_2))(x).$$

We conclude that to get properly defined constraints, we must choose an element $\boldsymbol{y} \in S(x_0)$ carefully.

5 An idea of Vessiot

In one of his last articles [14], Vessiot suggests to take the solution $y(w_1, w_2; x)$ of (7) with initial conditions

$$y(w_1, w_2; x_0) = w_1, \quad y_x(w_1, w_2; x_0) = w_2. \tag{9}$$

He proposes, in place of $(\Gamma_2, S(x_0))$, to introduce the following partial differential field extension

$$\mathbb{L} := \mathbb{C}(w_1, w_2)\left(x, \frac{\partial^{l+m+n} y(w_1, w_2; x)}{\partial x^l \partial w_1^m \partial w_2^n}\right)_{l,m,n=0,1,2,\ldots} \Big/ \mathbb{C}(w_1, w_2)(x) \tag{10}$$

with derivations $\partial/\partial x$, $\partial/\partial w_1$, $\partial/\partial w_2$, where $y(w_1, w_2; x)$ is the solution of (7) satisfying (9). His idea is that the partial differential field extension (10) gives us a kind of Galois closure of the initial ordinary differential field extension

$$\mathbb{C}(x, y(w_1, w_2; x), y_x(w_1, w_2; x))/\mathbb{C}(x)$$

with derivation d/dx. Let us check his idea with examples.

Example 5.1. Let us study the linear differential equation

$$y'' = p(x)y, \tag{11}$$

according to the idea of Vessiot, where $p(x)$ is a polynomial in x.

We take a solution $y(w_1, w_2; x)$ of (11) that satisfies the initial conditions (9) at the point $x = x_0$. We start from a differential field extension

$$\mathbb{C}(x, y(w_1, w_2; x), y'(w_1, w_2; x))/\mathbb{C}(x) \tag{12}$$

and show how Vessiot's idea leads us to a Galois extension. To describe the solution $y(w_1, w_2; x)$, we use two particular solutions $y_1(x)$, $y_2(x)$ of (11) such that

$$y_1(x_0) = 1, \quad y_1'(x_0) = 0, \quad y_2(x_0) = 0, \quad y_2'(x_0) = 1.$$

Thus we have

$$y(w_1, w_2; x) = w_1 y_1(x) + w_2 y_2(x).$$

Since $\partial y/\partial w_1 = y_1$, $\partial y/\partial w_2 = y_2$, we have

$$\begin{aligned}\mathbb{C}(w_1, w_2)&\left(x, \frac{\partial^{l+m+n} y(w_1, w_2; x)}{\partial x^l \partial w_1^m \partial w_2^n}\right)_{l,m,n,=0,1,2,\dots} \\ &= \mathbb{C}(w_1, w_2)(x, y_1(x), y_2(x), y_1'(x), y_2'(x))\end{aligned}$$

so that extension (10) is nothing but

$$\mathbb{C}(w_1, w_2)(x, y_1(x), y_2(x), y_1'(x), y_2'(x))/\mathbb{C}(w_1, w_2)(x)), \tag{13}$$

that is a Picard–Vessiot extension with derivations $\partial/\partial x$, $\partial/\partial w_1$, $\partial/\partial w_2$. So we started from the extension (12) of an ordinary differential field with derivation d/dx and, by the procedure of Vessiot, we arrived at the Picard–Vessiot extension (13). This shows that the procedure of Vessiot is really a normalization process of the given extension (12).

Example 5.2. The Riccati equation, given by

$$z' = -z^2 - p(x), \tag{14}$$

where $p(x)$ is a polynomial of x.

We consider the solution $z(w; x)$ of (14) with $z(w; x_0) = w$. We start from a differential field extension

$$\mathbb{C}(x, z(w; x), z_x(w; x))/\mathbb{C}(x) \tag{15}$$

with derivation d/dx. Let us see how Vessiot's idea works in this case. As is well-known the Riccati equation (14) is linearized by the linear equation

$$y'' = p(x)y. \tag{16}$$

We denote du/dx by u'. Using the notation of the previous example,

$$z(w; x) = \frac{\frac{d}{dx}(wy_2(x) + y_1(x))}{wy_2(x) + y_1(x)}. \tag{17}$$

We are going to determine the partial differential over-field

$$\mathbb{L} = \mathbb{C}(x, w)\left(\frac{\partial^{l+m} z(w; x)}{\partial x^l \partial w^m}\right)_{l,m=0,1,2,\dots} \Big/ \mathbb{C}(x, w)$$

with derivations $\partial/\partial x$ and $\partial/\partial w$.

Lemma 5.3. *We have* $y_i y_j \in \mathbb{L}$ *for* $1 \le i, j \le 2$.

Proof. Applying $\partial/\partial w$ to (17), since

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 1, \tag{18}$$

we get

$$\frac{1}{(wy_2 + y_1)^2} \in \mathbb{L} \tag{19}$$

and hence

$$(wy_2 + y_1)^2 = w^2 y_2^2 + 2wy_2 y_1 + y_1^2 \in \mathbb{L}. \tag{20}$$

Now we apply $\partial^2/\partial w^2$ to (20) to conclude that

$$y_2^2 \in \mathbb{L}. \tag{21}$$

Then it follows from (20) and (21) that

$$2wy_2 y_1 + y_1^2 \in \mathbb{L}. \tag{22}$$

Now differentiating (22) with respect to w, we get $y_1 y_2 \in \mathbb{L}$ so that consequently, by (22), $y_1^2 \in \mathbb{L}$. Hence we have seen

$$y_i y_j \in \mathbb{L} \quad \text{for } 1 \le i, j \le 2.$$ □

Lemma 5.4. *We have the following identities:*

$$(y_i y_j)' = y_i' y_j + y_i y_j' \quad \textit{for } 1 \le i, j \le 2.$$

Proof. This is just the Leibniz rule. □

Lemma 5.5. *We have*

$$\begin{aligned} (y_i y_j')' &= y_i' y_j' + p(x) y_i' y_j \qquad \textit{for } 1 \le i, j \le 2, \\ (y_i' y_j')' &= p(x)(y_i y_j' + y_i' y_j) \quad \textit{for } 1 \le i, j \le 2. \end{aligned}$$

Proof. These equalities follow from the Leibniz rule and (16). □

Proposition 5.6.

$$M := \mathbb{C}(x, w)(y_i y_j, y_i y_j', y_i' y_j')_{1 \le i, j \le 2} = \mathbb{L}.$$

Proof. It follows from (18), Lemmas 5.3, 5.4 and 5.5 that $y_i y_j$, $y_i y_j'$, $y_i' y_j'$ are in $\mathbb{L}$ so that $M \subset \mathbb{L}$. To show the inclusion $\mathbb{L} \subset M$, since M is closed under the partial derivations $\partial/\partial x$ and $\partial/\partial w$, it is sufficient to notice

$$z(w; x) = \frac{wy_2' + y_1'}{wy_2 + y_1} = \frac{wy_2' y_1 + y_1' y_1}{wy_2 y_1 + y_1 y_1} \in M$$

by (17). □

Now by Lemmas 5.4, 5.5, with respect to the derivation $\partial/\partial x$, the elements $y_i y_j$, $y_i y_j'$ and $y_i' y_j'$ for $1 \leq i, j \leq 2$ satisfy a homogeneous system of linear equations with coefficients in $\mathbb{C}(x)$ and these elements are constant with respect to $\partial/\partial w$. So $M = \mathbb{L}/\mathbb{C}(x, w)$ is a Picard–Vessiot extension by Proposition 5.6. Namely starting from the differential field extension (15) with derivation d/dx, the procedure of Vessiot leads us to a Galois extension $\mathbb{L}/\mathbb{C}(w)(x)$ with derivations d/dx, $\partial/\partial w_1$, $\partial/\partial w_2$, revealing the hidden PSL_2-symmetry of the extension (15).

6 Our theory

In the previous section we have seen by two examples that the idea of Vessiot seems to work. Now let us realize the idea of Vessiot on a rigorous algebraic framework; in other words, let us explain how, algebraically, considering a differential equation may be viewed as equivalent to considering a differential algebra extension. We start by briefly recalling basic notions. All the rings that we consider are commutative with 1 and contain $\mathbb{Q}$. A derivation on a ring R is a map $\delta\colon R \to R$ satisfying (i) $\delta(a + b) = \delta(a) + \delta(b)$ and (ii) $\delta(ab) = \delta(a)b + a\delta(b)$ for every $a, b \in R$. A differential ring (R, δ) consists of a ring R and a derivation $\delta\colon R \to R$. Let P be a subring of the differential ring (R, δ) closed under the derivation δ so that (P, δ) is a differential ring. We say that (P, δ) is a differential subring of (R, δ) or $(R, \delta)/(P, \delta)$ is a differential ring extension. For a subset $Z \subset R$, we denote by $P\{Z\}$ the differential subring generated by the subset Z over P, which is the smallest differential subalgebra of (R, δ) containing both P and Z.

So far we treated a differential ring with a derivation. We have to treat also differential rings $(R, \{\delta_1, \delta_2, \dots, \delta_n\})$ with several derivations $\delta_1, \delta_2, \dots, \delta_n$ such as $(\mathbb{C}(x, y), \{\partial/\partial x, \partial/\partial y\})$. We always assume that the derivations $\delta_i' s$ are mutually commutative. We call the differential ring $(R, \{\delta_1, \delta_2, \dots, \delta_n\})$ a partial differential ring if $n \geq 2$ or an ordinary differential ring if $n = 1$. We call an element a of R constant if $\delta_i(a) = 0$ for $1 \leq i \leq n$. The set C_R of constants of R forms a subring of R called the ring of constants.

Let K be the rational function field $\mathbb{C}(x)$ of one variable x so that $(K, d/dx)$ is a differential field. Let y, y' be variables over K. We denote by L the rational function field $\mathbb{C}(x, y, y') = K(y, y')$ of three variables with coefficients in $\mathbb{C}$. Let $F(x, y, y') \in \mathbb{C}[x, y, y']$. We study a differential equation $y'' = F(x, y, y')$. Algebraically we define a derivation on the field L such that L/K is a differential field extension. Namely we set

$$\delta(x) = 1, \quad \delta(\mathbb{C}) = 0, \quad \delta(y) = y', \quad \delta(y') = F(x, y, y')$$

and extend it to the derivation $\delta\colon L \to L$ by linearity (i), the Leibniz rule (ii) and by the formula $\delta(a/b) = (\delta(a)b - a\delta(b))/b^2$ for $a, b \neq 0 \in \mathbb{C}[x, y, y']$.

Let (R, δ) be a differential ring. We often refer to the differential ring R without making the derivation δ precise. For this reason, when we deal with the ring structure R of the differential ring (R, δ) or the ring R itself, we denote it by $R^\natural$. In other words, $^\natural$ means to forget the derivation δ.

X being a variable over the ring R, we define the universal Taylor morphism

$$i\colon R \to R^\natural[[X]]$$

by setting

$$i(a) = \sum_{n=0}^{\infty} \frac{1}{n!}\delta^n(a)X^n \tag{23}$$

for $a \in R$. Namely $i(a)$ is the formal Taylor expansion of the element $a \in R$. The universal Taylor morphism $(R, \delta) \to (R^\natural[[X]], d/dX)$ is a differential algebra morphism. Let us see what the universal Taylor morphism is by examples.

Example 6.1. Let x be a variable over $\mathbb{C}$ and $K := (\mathbb{C}(x), d/dx)$. Let y be a variable over $\mathbb{C}(x)$ and set $L := \mathbb{C}(x, y)$ so that L is the rational function field of two variables x, y. We extend the derivation d/dx of the field $K = \mathbb{C}(x)$ to L by setting $\delta(y) = y$. Analytically we consider the differential equation $y' = y$. It follows from the definition (23) of the universal Taylor morphism $i\colon L \to L^\natural[[X]]$ that we have

$$i(y) = \sum_{n=0}^{\infty} \frac{1}{n!}\delta^n(y)X^n = \sum_{n=0}^{\infty} \frac{1}{n!} yX^n = y \exp X.$$

So $i(y) = Y(y; X)$ is the solution of the differential equation

$$\frac{d}{dX}Y(y; X) = Y(y; X),$$

taking the initial condition y at $X = 0$, namely at $i(x) = x + X = x$, which means at the generic point of the complex plane $\mathbb{C}$. This illustrates that the universal Taylor morphism (23) gives a Taylor expansion of the solution of the differential equation $y' = y$ taking the initial condition y at the generic point of the complex plane $\mathbb{C}$.

Example 6.2. Let $K = (\mathbb{C}(x), d/dx)$ as above and $L = \mathbb{C}(x, y, y')$, y, y' being variables over K. So L is the rational function field of three variables x, y, y' with coefficients in $\mathbb{C}$. Let $F(x, y, y') = 6y^2 + x$. We define the derivation δ on L as we explained above. Namely we consider the first Painlevé equation $y" = 6y^2 + x$. What is the image $i(y)$ of the element $y \in L$ by the universal Taylor morphism $i\colon L \to L^\natural[[X]]$? It follows from the definition (23) that

$$i(y) = y + y'X + \frac{1}{2!}\delta(y')X^2 + \cdots = y + y'X + \frac{1}{2}(6y^2 + x)X^2 + \cdots \in L^\natural[[X]]. \tag{24}$$

Since the universal Taylor morphism $i\colon L \to L^\natural[[X]]$ is a differential algebra morphism and since $\delta^2(y) = \delta(y') = 6y^2 + x$, if we notice $i(x) = x + X \in L^\natural[[X]]$ by definition, then $i(y) = Y(y, y'; X) \in L^\natural[[X]] = \mathbb{C}(x, y, y')[[X]]$ satisfies

$$\frac{\partial^2 Y}{\partial X^2} = 6Y^2 + i(x) = 6Y^2 + x + X. \tag{25}$$

It follows from (24) that

$$Y(y, y'; 0) = y, \quad \partial Y(y, y'; 0)/\partial X = y'. \tag{26}$$

This shows that $i(y) = Y(y, y', X)$ is the solution of the first Painlevé equation (25) taking the initial conditions (26) at the generic point $i(x) = x$ of the complex plane.

These two examples show that we can apply the idea of Vessiot to the solutions $Y(y; X)$ and $Y(y, y'; X)$ respectively.

Whereas we are working in the ring $L^\natural[[X]]$, we are going to generate fields. So we replace the ring $L^\natural[[X]]$ by its quotient field, that is, the field $L^\natural[[X]][X^{-1}]$ of Laurent series. We regard $i(K)$ and $i(L)$ as subfields of the field $L^\natural[[X]][X^{-1}]$. First we analyze Example 6.1. According to Vessiot, we consider the differential subfield $\mathcal{L}'$ of $L^\natural[[X]][X^{-1}]$ generated by $i(L)$ with respect to the derivations $\partial/\partial y$ and d/dX. Here $\partial/\partial y$ is the derivation

$$\partial/\partial y\colon L^\natural = \mathbb{C}(x, y) \to L^\natural = \mathbb{C}(x, y)$$

operating on the coefficients of the field $L^\natural[[X]][X^{-1}]$. More directly, $\partial/\partial y$ is the derivation

$$\partial/\partial y\colon L^\natural[[X]][X^{-1}] = \mathbb{C}(x, y)[[X]][X^{-1}] \to L^\natural[[X]][X^{-1}] = \mathbb{C}(x, y)[[X]][X^{-1}].$$

We have to get a kind of Galois extension $\mathcal{L}'/i(K)$. This is almost correct. The problem is that we start from the differential field extension $L = K(y)/K$ so that the differential field extension $\mathcal{L}'/i(K)$ must be canonically determined from the extension L/K. The extension $\mathcal{L}'/i(K)$ constructed above depends, however, on the choice of the generator y of L over K. The choice of y determines the derivation $\partial/\partial y\colon \mathbb{C}(x, y) \to \mathbb{C}(x, y)$ which is non-canonical. To avoid this non-canonical choice, we consider all the derivations

$$\mathrm{Der}(\mathbb{C}(x, y)/\mathbb{C}(x)) = \mathbb{C}(x, y)\frac{\partial}{\partial y},$$

the right-hand side being the set of derivations $\delta\colon \mathbb{C}(x, y) \to \mathbb{C}(x, y)$ such that $\delta(\mathbb{C}(x)) = 0$. Coming back to the definition, we define $\mathcal{L}$ to be the differential subfield of $L^\natural[[X]][X^{-1}]$ generated by $i(L)$ and $L^\natural$ with respect to the derivations $\partial/\partial y$ and d/dX. Similarly we denote by $\mathcal{K}$ the differential subfield of $L^\natural[[X]][X^{-1}]$ generated by $i(K)$ and $L^\natural$ with respect to the derivations $\partial/\partial y$ and d/dX. Since the subfield $i(K).L^\natural = L^\natural(X)$ generated by $i(K)$ and $L^\natural$ in $L^\natural[[X]][X^{-1}]$ is closed under the derivations $\partial/\partial y$ and d/dX we have $\mathcal{L} = L^\natural(X) = \mathbb{C}(x, y)(X)$, equipped with the derivations $\partial/\partial y$ and d/dX. Similarly, since $Y(y; X) = y\exp(X)$,

$$\mathcal{L} = \mathcal{K}(y\exp(X)) = \mathcal{K}(\exp(X)) = \mathbb{C}(x, y, X)(\exp(X)),$$

equipped with the derivations $\partial/\partial y$ and d/dX. Hence the constructed partial differential field extension $\mathcal{L}/\mathcal{K}$ is $\mathbb{C}(x, y)(X)(\exp(X))/\mathbb{C}(x, y)(X)$ with derivations $\partial/\partial y$ and d/dX that looks truly like a Galois extension. So in Example 6.1, we started from the Picard–Vessiot extension $\mathbb{C}(x, y)/\mathbb{C}(x)$ with $\delta(x) = 1$ and $\delta y = y$ and we are led to the partial differential field extension

$$\mathcal{L} = \mathbb{C}(x, y, X)(\exp(X)) \quad \text{where } \mathcal{K} = \mathbb{C}(x, y)(X).$$

The Galois group $\mathcal{G}$ of the differential equation in this framework should be the group $\mathrm{Aut}(\mathcal{L}/\mathcal{K})$ of automorphisms of the partial differential field $\mathcal{L}$ that leave every element of the subfield $\mathcal{K}$ fixed. So the group $\mathcal{G}$ is isomorphic to the multiplicative group $\mathbb{C}(x)^{\times}$.

Remarks 6.3. (i) The field $C_{\mathcal{L}} := \{a \in \mathcal{L} \subset \mathbb{C}(x, y)[[X]][X^{-1}] \mid \partial a/\partial y = da/dX = 0\}$ of constants of the partial differential field $\mathcal{L}$ is $\mathbb{C}(x) \subset L^{\natural} = \mathbb{C}(x, y)$.

(ii) Since the differential equation $y' = y$ is a linear ordinary differential equation, we can speak of the Galois group G in the sense of Picard–Vessiot. The Galois group G in this sense is the multiplicative group $\mathbb{C}^{\times}$.

(iii) The difference between the Galois group in (ii) and the Galois group $\mathcal{G}$ is reasonable, if we recall that in the realization of Vessiot's idea, we used the universal Taylor extension, that is the Taylor expansion at the generic point of the complex plane.

Now we pass to Example 6.2, the first Painlevé equation. In this case the above argument shows that $\mathcal{K} = L^{\natural}(X) = \mathbb{C}(x, y, y')(X)$ with derivations $\partial/\partial y$, $\partial/\partial y'$ and d/dX. Let $\mathcal{S}$ be the differential subalgebra of $L^{\natural}[[X]]$ generated by $i(L)$ and $L^{\natural}$. So $\mathcal{S} = L^{\natural}\{Y(y, y'; X)\}$ with partial derivations $\partial/\partial y$, $\partial/\partial y'$ and d/dX and $\mathcal{L}$ is the quotient field of $\mathcal{S}$. We can show

$$\frac{D(Y(y, y'; x), \partial Y(y, y'; x)/\partial X)}{D(y, y')} = \begin{vmatrix} \frac{\partial Y}{\partial y} & \frac{\partial Y}{\partial y'} \\ \frac{\partial^2 Y}{\partial X \partial y} & \frac{\partial^2 Y}{\partial X \partial y'} \end{vmatrix} = 1. \tag{27}$$

An affirmative answer to the following question is expected. See [13], 4.5, Example 3.

Question 6.4. *Does the relation* (27) *differentially generate the ideal in S of all the algebraic relations among the derivatives*

$$\frac{\partial^{l+m+n} Y(y, y'; X)}{\partial X^l \partial^m \partial^n}, \quad l, m, n = 0, 1, 2, \ldots,$$

with coefficients in $\mathcal{K} = L^{\natural}(X)$?

Now the Galois group of the first Painlevé equation in this framework is the group of all the automorphisms of the partial differential field $\mathcal{L}$ leaving every element of $\mathcal{K}$ fixed. If Question 6.4 has an affirmative answer, the transcendence degree of the field extension $\mathcal{L}/\mathcal{K}$ is infinite and hence the Galois group is infinite dimensional. Describing an infinite dimensional group belongs to a technical part of our theory. We use Lie pseudo-groups, that is, a set of coordinate transformations defined by algebraic partial differential equations, containing the identity, closed under the composition and taking the inverse. A concrete example will help us to understand what it is.

Example 6.5. Let us consider the set G of 1-dimensional coordinate transformations $\varphi(y)$ satisfying the differential equation

$$\{\varphi; y\} = \left(\frac{d^3\varphi}{dy^3}\right) \Big/ \left(\frac{d\varphi}{dx}\right) - \frac{3}{2}\left[\frac{d^2\varphi}{dy^2}\Big/\frac{d\varphi}{dy}\right]^2 = 0.$$

The left-hand side is known as the Schwarzian derivative of the function $\varphi(y)$. We know that the solution $\varphi(y)$ is written in the form

$$\varphi(y) = \frac{ay+b}{cy+d}, \quad a, b, c, d \in \mathbb{C} \text{ with } ad - bc \neq 0$$

so that the set G contains the identity and is closed under the composition and taking the inverse. In fact G is a group in this example.

Example 6.6. Let us consider the set G of all coordinate transformations $(y_1, y_2) \to (\varphi_1(y_1, y_2), \varphi_2(y_1, y_2))$ satisfying an algebraic partial differential equation

$$\frac{D(\varphi_1, \varphi_2)}{D(y_1, y_2)} = \begin{vmatrix} \frac{\partial\varphi_1(y_1,y_2)}{\partial y_1} & \frac{\partial\varphi_1(y_1,y_2)}{\partial y_2} \\ \frac{\partial\varphi_2(y_1,y_2)}{\partial y_1} & \frac{\partial\varphi_2(y_1,y_2)}{\partial y_2} \end{vmatrix} = 1.$$

Then the set G is defined by the algebraic partial differential equation, is closed under the composition and taking the inverse, and G contains the identity.

If Question 6.4 is affirmatively answered, then we can conclude that the Galois group of the first Painlevé equation is the Lie pseudo-group of Example 6.6. To be more precise, the Galois group that is the automorphism group of the partial differential field extension$\mathcal{L}/\mathcal{K}$ coincides with the group of automorphisms induced by the transformations

$$(Y(y, y'; X), \partial Y(y, y'; X)/\partial X) \mapsto$$
$$(Y(\varphi_1(y, y'), \varphi_2(y, y'); X), \partial Y(\varphi_1(y, y'), \varphi_2(y, y'); X)/\partial X)$$

satisfying

$$\frac{D(\varphi_1, \varphi_2)}{D(y_1, y_2)} = \begin{vmatrix} \frac{\partial\varphi_1(y,y')}{\partial y} & \frac{\partial\varphi_1(y,y')}{\partial y'} \\ \frac{\partial\varphi_2(y,y')}{\partial y} & \frac{\partial\varphi_2(y_1,y_2)}{\partial y'} \end{vmatrix} = 1.$$

Now we sketch the general definition of Galois group. Let L/K be an ordinary differential field extension such that $L^\natural$ is finitely generated over $K^\natural$ as an abstract field extension. Let n be the transcendence degree of $L^\natural$ over $K^\natural$. Let us denote by $\mathrm{Der}(L^\natural/K^\natural)$ the set

$$\{\delta\colon L \to L \mid \delta \text{ is a derivation such that } \delta(K^\natural) = 0\}.$$

Then $\mathrm{Der}(L^\natural/K^\natural)$ is an $L^\natural$-vector space of dimension n. It is well-known that there exist n mutually commutative derivations

$$\delta_1, \delta_2, \ldots, \delta_n \in \mathrm{Der}(L^\natural/K^\natural)$$

such that

$$\mathrm{Der}(L^\natural/K^\natural) = L^\natural\delta_1 \oplus L^\natural\delta_2 \oplus \cdots \oplus L^\natural\delta_n.$$

We denote the partial differential field $(L^\natural, \{\delta_1, \delta_2, \ldots, \delta_n\})$ by $L^\sharp$. Let $i\colon L \to L^\natural[[X]]$ be the universal Taylor morphism. We let the derivations $\delta_1, \delta_2, \ldots, \delta_n$ operate on the coefficients in $L^\natural[[X]]$. In other words we regard the universal Taylor morphism as a map $i\colon L \to L^\sharp[[X]]$. Then $(L^\sharp[[X]], \{d/dX, \delta_1, \delta_2, \ldots, \delta_n\})$ is a partial differential ring. Let $\mathcal{K}$ be the partial differential subfield generated by $i(K)$ and $L^\sharp$ in the partial differential field $L^\sharp[[X]][X^{-1}]$. Similarly let $\mathcal{L}$ be the partial differential field of the partial differential field $L^\sharp[[X]][X^{-1}]$ generated by $i(L)$ and $L^\sharp$. The Galois group is the automorphism group $\mathrm{Aut}(\mathcal{L}/\mathcal{K})$ of the partial differential field extension, which is a Lie pseudo-group. The fact that the group $\mathrm{Aut}(\mathcal{L}/\mathcal{K})$ of symmetries is big or that the extension $\mathcal{L}/\mathcal{K}$ is a kind of Galois extension is characterized in terms of principal homogeneous space. See [10] and [13].

7 Relation with P. Cassidy and M. Singer's parameterized Picard–Vessiot theory

We discussed the comparison of Malgrange's Galois theory of foliations and ours in [13]. As Cassidy–Singer [3] proposed at this conference a Galois theory of parameterized Picard–Vessiot theory, we clarify the relationship of the Cassidy–Singer theory with our general theory. Namely we show that parameterized Picard–Vessiot extensions appear naturally in our general differential Galois theory.

Let $A(x, t_1, t_2, \ldots, t_m)$ be an $n \times n$-matrix with entries in

$$\mathbb{C}(x, t_1, t_2, \ldots, t_m).$$

We consider a linear ordinary differential equation

$$\frac{\partial y}{\partial x} = A(x, t_1, t_2, \ldots, t_m)y \tag{28}$$

parameterized by $t_1, t_2, \ldots, t_m$. Here $y = (y_{ij})$ is an $n \times n$-matrix with $\det y \neq 0$ Analytically one may imagine that $y_{ij} = y_{ij}(x, t_1, t_2, \ldots, t_m)$ is an analytic function of $x, t_1, t_2, \ldots, t_m$ near a particular point

$$(x, t_1, t_2, \ldots, t_m) = (x_0, t_{10}, t_{20}, \ldots, t_{m0}) \in \mathbb{C}^{m+1}$$

for $1 \le i, j \le n$.

In terms of differential field extension, let $k := \mathbb{C}(x, t_1, t_2, \ldots, t_m)$, $\delta_0 = \partial/\partial x$, $\delta_1 = \partial/\partial t_1, \ldots, \delta_m = \partial/\partial t_m$ and $\Delta := \{\delta_0, \delta_1, \ldots, \delta_m\}$ so that (k, Δ) is a partial differential field. Let

$$K := k\langle y_{ij}\rangle_{1\le i,j\le n} = k(\delta_0^{l_0}\delta_1^{l_1}\delta_2^{l_2}\dots\delta_m^{l_m} y_{ij})_{1\le i,j\le n,(l_0,l_1,l_2,\dots,l_m)\in\mathbb{N}^{m+1}},$$

that is, the partial differential field generated over k by the y_{ij}'s with derivations $\delta_0, \delta_1, \dots, \delta_m$. So $(K, \Delta)/(k, \Delta)$ is a partial differential field extension that Cassidy and Singer call a parameterized Picard–Vessiot extension, or PPV-extension for short. The Galois group $\mathrm{Gal}_\Delta(K/k)$ of the PPV-extension K/k is the group of k-automorphisms of the partial differential field extension.

To be more concrete, a transformation $y = (y_{ij}) \mapsto \sigma(y) = (\sigma(y_{ij}))$ induces a transformation

$$\frac{\partial^{l_0+l_1+\dots+l_m} y_{ij}}{\partial x^{l_0}\partial t_1^{l_1}\dots\partial t_m^{l_m}} \mapsto \frac{\partial^{l_0+l_1+\dots+l_m}\sigma(y_{ij})}{\partial x^{l_0}\partial t_1^{l_1}\dots\partial t_m^{l_m}}$$

for $1 \le i, j \le n$ and for $(l_0, l_1, \dots, l_m) \in \mathbb{N}^{m+1}$. The Galois group $\mathrm{Gal}_\Delta(K/k)$ of the parameterized Picard–Vessiot equation (28) by Cassidy and Singer is the group of transformations $y = (y_{ij}) \mapsto \sigma(y) = (\sigma(y_{ij}))$ that leave all elements of $\mathbb{C}(x, t_1, t_2, \dots, t_m)$ invariant and preserve all algebraic relations among the derivatives

$$\frac{\partial^{l_0+l_1+\dots+l_m} y_{ij}}{\partial x^{l_0}\partial t_1^{l_1}\dots\partial t_m^{l_m}}, \quad (l_0, l_1, \dots, l_m) \in \mathbb{N}^{m+1},\ 1 \le i, j \le n$$

with coefficients in $\mathbb{C}(x, t_1, t_2, \dots, t_m)$. To have a clear image of the Galois group $\mathrm{Gal}_\Delta(K/k)$, let us study an example.

Example 7.1 (Cassidy and Singer [3], Example 3.1). Let

$$\Delta := \{\partial/\partial x, \partial/\partial t\} = \{\partial_x, \partial_t\}, \quad k := \mathbb{C}(x, t), \quad K := \mathbb{C}(x, t, x^t, \log x)$$

so that (K, Δ) is a partial differential field extension of (k, Δ). The functions $y := x^t$ and $\log x$ satisfy the differential equations

$$\partial_x y = \frac{t}{x} y, \tag{29}$$

$$\partial_t y = (\log x) y, \tag{30}$$

$$\partial_x \log x = \frac{1}{x}, \tag{31}$$

$$\partial_t \log x = 0, \tag{32}$$

that characterize the partial differential field extension K/k. Moreover, $\log x$ and $y = x^t$ are algebraically independent over $\mathbb{C}(x, t)$. Now let $\sigma \in \mathrm{Gal}_\Delta(K/k)$. Cassidy and Singer [3] show that these differential equations imply that $\sigma(y) = \exp(ct + c')y$ with $c, c' \in \mathbb{C}$. Namely the Galois group $\mathrm{Gal}_\Delta(K/k)$ is isomorphic to the multiplicative group $\{\exp(ct + c') \mid c, c' \in \mathbb{C}\}$. We notice here that the group $\{\exp(ct + c') \mid c, c' \in \mathbb{C}\}$. is a Lie pseudo-group. In fact,

$$\Big\{\exp(ct + c') \mid c, c' \in \mathbb{C}\} = \{t \mapsto a(t) \mid 0 = (\log a)'' := \Big(\frac{a'}{a}\Big)'\Big\}.$$

Now we study by our general theory the linear ordinary differential equation

$$\frac{dy}{dx} = \frac{t}{x}y,$$

that is, the differential equation (29) with coefficients in $\mathbb{C}(x, t)$. In terms of ordinary differential field extension, we start from $(\mathbb{C}(t, x), d/dx)$ and consider an ordinary differential field extension

$$(\mathbb{C}(x, t, y), d/dx)/(\mathbb{C}(x, t), d/dx) \tag{33}$$

such that y is transcendental over $\mathbb{C}(x, t)$ and we extend the derivation d/dx of $\mathbb{C}(x, t)$ by setting $dy/dx := ty/x$ so that (33) is a Picard–Vessiot extension with Galois group $\mathbb{C}(t)^\times$, the multiplicative group of the non-zero elements of the field $\mathbb{C}(t)$ of constants of the ordinary differential field $(\mathbb{C}(x, t), d/dx)$. We are interested in the ordinary differential field extension

$$(\mathbb{C}(x, t, y), d/dx)/(\mathbb{C}(x), d/dx) \tag{34}$$

which has a non-trivial constant field extension. Setting

$$L := (\mathbb{C}(x, t, y), d/dx) \quad \text{and} \quad K := (\mathbb{C}(x), d/dx),$$

let $i\colon L \to L^\natural[[X]][X^{-1}]$ be the universal Taylor morphism so that

$$Y(t, y; X) := i(y) = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{d^n y}{dx^n}X^n. \tag{35}$$

To determine the series $Y(t, y; X)$, let us notice that

$$\frac{dy}{dx} = \frac{t}{x}y$$

by definition, and successively by calculation we get

$$\frac{d^2y}{dx^2} = \frac{t(t-1)}{x^2}y, \quad \frac{d^3y}{dx^3} = \frac{t(t-1)(t-2)}{x^3}y, \dots.$$

In general we have

$$\frac{d^n y}{dx^n} = \frac{t(t-1)\dots(t-n+1)}{x^n}y$$

so that by (35)

$$Y(t, y; X) := y(1 + \frac{X}{x})^t, \tag{36}$$

where

$$\left(1 + \frac{X}{x}\right)^t = \sum_{n=0}^{\infty} \frac{t(t-1)\dots(t-n+1)}{n!}\ \frac{X}{x}^n$$

by definition.

We have two derivations $\partial/\partial t$, $\partial/\partial y$ of $L^{\natural} = \mathbb{C}(x, t, y)$. These two derivations span the $L^{\natural}$-vector space $\mathrm{Der}(L^{\natural}/K^{\natural})$ so that

$$\mathrm{Der}(L^{\natural}/K^{\natural}) = L^{\natural}\frac{\partial}{\partial t} \oplus L^{\natural}\frac{\partial}{\partial y}.$$

We introduce a partial differential field $L^{\sharp} := (L^{\natural}, \{\partial/\partial t, \partial/\partial y\})$.

The expression (36) suggests us to replace Y by $y^{-1}Y$. So we denote $y^{-1}Y(t, y; X)$ by $Z(t, y; X))$. Then

$$\mathcal{L} = i(K) \cdot L^{\sharp}\langle Y(t, y; X)\rangle = i(k) \cdot L^{\sharp}\langle Z(t, y;\ X)\rangle$$

and Z satisfies the following differential equations:

$$\frac{\partial Z(t, y; X)}{\partial X} = \frac{t}{x + X} Z(t, y; X), \tag{37}$$

$$\frac{\partial Z}{\partial y} = 0, \tag{38}$$

$$\frac{\partial Z}{\partial t} = \log\left(1 + \frac{X}{x}\right) Z, \tag{39}$$

$$\frac{\partial}{\partial t}\log\left(1 + \frac{X}{x}\right) = 0, \tag{40}$$

$$\frac{\partial}{\partial X}\log\left(1 + \frac{X}{x}\right) = \frac{1}{x + X}. \tag{41}$$

We notice here that $i(x) = x + X$. The differential equations (37), (38),…, (41) show that we are in the situation of Example 7.1. The argument of Cassidy and Singer shows that our Galois group of extension (34), which is the automorphism group of the partial differential field extension $\mathcal{L}/\mathcal{K}$ coincides with Cassidy and Singer's.

Remark 7.2. Let (x, t) be the natural coordinate system on $\mathbb{C}^2$ so that x, t are variables over $\mathbb{C}$. Let U be a sufficiently small connected open neighborhood of the point $(1, 0) \in \mathbb{C}^2$. We work in the differential field $(\mathcal{U}, \partial/\partial x)$ of all the meromorphic functions on U. Let $y(x, t)$ be the solution of the differential equation

$$\frac{\partial y(x, t)}{\partial x} = \frac{t}{x} y(x, t),$$

holomorphic on U with $y(1, t) = t$ for $(1, t) \in U$. Setting $x^t := y(x, t)$, we have $x^t \in \mathcal{U}$. Let $K = (\mathbb{C}(x), d/dx)$, which is naturally considered as a differential subfield of $\mathcal{U}$. Let $\zeta(t)$ be the Riemann zeta function, which is a meromorphic function in t. So we can consider $\zeta(t) \in \mathcal{U}$. Let us set

$$L' := \mathbb{C}(x, t, x^t), \quad L'' := \mathbb{C}(x, t, x^t\zeta(t))$$

so that $(L', \partial/\partial x)$ and $(L'', \partial/\partial x)$ are differential subfields of $\mathcal{U}$. The ordinary differential field extension L/K (34) is isomorphic to L'/K and L''/K because we consider only the derivation $\partial/\partial x$. So if we construct the partial differential field extensions $\mathcal{L}'/\mathcal{K}$ and $\mathcal{L}''/\mathcal{K}$ respectively from (L', K) and (L'', K) by the general procedure, then the partial differential field extensions $\mathcal{L}/\mathcal{K}$, $\mathcal{L}'/\mathcal{K}$ and $\mathcal{L}''/\mathcal{K}$ are canonically isomorphic. Therefore, if we denote our Galois group by InfGal, then InfGal $(\mathbb{C}(x, t, y)/\mathbb{C}(x))$ is isomorphic to both

$$\mathrm{InfGal}\left(\mathbb{C}(x, t, x^t)/\mathbb{C}(x)\right) \quad \text{and} \quad \mathrm{InfGal}\left(\mathbb{C}(x, t, x^t\zeta(t))/\mathbb{C}(x)\right).$$

8 What is general differential Galois theory good for?

Galois group is an invariant attached to an ordinary differential field extension or to an algebraic ordinary differential equation. Since this is the only one existent invariant, it is easy to imagine its theoretical importance.

We know that the first Painlevé equation is not reducible to the classical functions. If we compare functions to stars, irreducibility means that a solution of the first Painlevé equation is not observable by a classical telescope. Calculation of the Galois group means, however, to observe the stars by a newly invented method. So it will bring us more information about the star than the un-observablity theorem. But it is not easy for the moment to calculate the Galois group for the first Painlevé equation.[1]

There is also a very interesting application of infinite dimensional differential Galois theory to monodromy preserving deformation due to Drach (cf. [2], [12]).

9 Can we calculate it?

For a linear ordinary differential equation, or for the Picard–Vessiot theory, there are computational studies. So we can calculate the Galois group in this case. Kolchin's theory is wider and it involves not only linear algebraic groups but also general algebraic groups, e.g. Abelian varieties. There are examples of calculations. Cassidy and Singer's PPV-extension is related with Picard–Vessiot theory and so we have non-trivial examples.

If we go beyond Kolchin's theory, it is very difficult to determine the Galois group. There is no non-trivial example except for the example related with monodromy preserving deformation [2], [12], [13].

[1] In the meantime (the present article was written in 2004), after having succeeded in calculating the Galois group of the first Painlevé equation in 2005 (*C. R. Math. Acad. Sci. Paris* 343 (2) (2006), 95–98), Guy Casale recently determined the Galois group of the Picard solution of the sixth Painlevé equation. In the latter case the solution is in general not classical, but the Galois group is finite dimensional. We can observe this phenomenon only by general differential Galois theory. So this fact enhances the value of general differential Galois theory.

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List of Participants

Jonathan AÏDAN (Institut Mathématique de Jussieu, Paris)
Yves ANDRÉ (École Normale Supérieure, Paris)
Dmitrii V. ANOSOV (Steklov Mathematical Institute, Moscow)
Michèle AUDIN (Université Louis Pasteur, Strasbourg)
Werner BALSER (Universität Ulm)
Moulay BARKATOU (Université de Limoges)
Pierre BAUMANN (Université Louis Pasteur, Strasbourg)
Maint BERKENBOSCH (Universität Heidelberg)
Daniel BERTRAND (Université Pierre et Marie Curie, Paris)
Philip BOALCH (École Normale Supérieure, Paris)
Delphine BOUCHER (Université de Rennes)
Louis BOUTET DE MONTVEL (Université Pierre et Marie Curie, Paris)
Mikhail BUTUZOV (Moscow State University)
Damien CALAQUE (Université Louis Pasteur, France)
Élie COMPOINT (Université de Lille)
Eduardo COREL (Institut Mathématique de Jussieu, Paris)
Michael DETTWEILER (Universität Heidelberg)
Antoine DOUAI (Université de Nice)
Boris DUBROVIN (SISSA, Trieste)
Anne DUVAL (Université de Lille)
Guillaume DUVAL (University of Valladolid)
Benjamin ENRIQUEZ (Université Louis Pasteur, Strasbourg)
Frédéric FAUVET (Université Louis Pasteur, Strasbourg)
Giovanni FELDER (ETH, Zürich)
Augistin FRUCHARD (Université de Haute Alsace, Mulhouse)
Agnès GADBLED (Université Louis Pasteur, Strasbourg)
Philippe GAILLARD (Université de Rennes)
Lubomir GAVRILOV (Université Paul Sabatier, Toulouse)
Alexey GLUTSYUK (École Normale Supérieure de Lyon)
Valentina GOLUBEVA (State Pedagogical Institute, Kolomna)
Renar GONTSOV (Moscow State University)
Davide GUZZETTI (RIMS, Kyoto)
Charlotte HARDOUIN (Institut Mathématique de Jussieu, Paris)
Julia HARTMANN (Universität Heidelberg)
Yulij ILYASHENKO (Cornell University, Ithaca)
Elisabeth ITS (IUPUI, Indianapolis)
Alexander ITS (IUPUI, Indianapolis)
Christian KASSEL (Université Louis Pasteur, Strasbourg)
Victor KATSNELSON (Weizmann Institute, Rehovot)
Viatcheslav KHARLAMOV (Université Louis Pasteur, Strasbourg)
Hironobu KIMURA (Kumamoto University)
Victor KLEPTSYN (École Normale Supérieure de Lyon)
Martine KLUGHERTZ (Université Paul Sabatier, Toulouse)
Vladimir KOSTOV (Université de Nice)
Nicolas LE ROUX (Université de Limoges)

Vladimir LEKSIN (State Pedagogical Institute, Kolomna)
Anton LEVELT (University of Nijmegen)
Michèle LODAY-RICHAUD (Université d'Angers)
Frank LORAY (Université de Rennes)
Stéphane MALEK (Université de Lille)
Bernard MALGRANGE (Institut Joseph Fourier, Grenoble)
Étienne MANN (Université Louis Pasteur, Strasbourg)
Jean-François MATTÉI (Université Paul Sabatier, Toulouse)
Heinrich MATZAT (Universität Heidelberg)
Marta MAZZOCCO (University of Cambridge)
Claude MITSCHI (Université Louis Pasteur)
Masatake MIYAKE (Nagoya University)
Claire MOURA (Laboratoire Émile Picard, Toulouse)
Robert MOUSSU (Université de Dijon)
Thomas OBERLIES (Universität Heidelberg)
Yousouke OHYAMA (Osaka University)
Kazuo OKAMOTO (University of Tokyo)
Emmanuel PAUL (Laboratoire Émile Picard, Toulouse)
Christian PESKINE (Centre National de la Recherche Scientifique, Paris)
Frédéric PHAM (Université de Nice)
Vladimir POBEREZHNY (Steklov Mathematical Institute, Moscow)
Jean-Pierre RAMIS (Université Paul Sabatier, Toulouse)
Stefan REITER (Universität Darmstadt)
Vladimir ROUBTSOV (Université d'Angers)
Claude SABBAH (École Polytechnique, Palaiseau)
Tewfik SARI (Université de Haute Alsace, Mulhouse)
Jacques SAULOY (Université Paul Sabatier, Toulouse)
David SAUZIN (Institut de Mécanique Céleste et de Calcul des Éphémérides, Paris)
Reinhard SCHÄFKE (Université Louis Pasteur, Strasbourg)
Hocine SELLAMA (Université Louis Pasteur, Strasbourg)
Michael SINGER (North Carolina State University, Raleigh)
Catherine STENGER (Université de La Rochelle)
Laurent STOLOVITCH (Université Paul Sabatier, Toulouse)
Konstantin STYRKAS (Max Planck Institut, Bonn)
Loïc TEYSSIER (Université Louis Pasteur, Strasbourg)
Jean THOMANN (Université Louis Pasteur, Strasbourg)
Frédéric TOUZET (Université de Rennes)
Albert TROESCH (Université Louis Pasteur, Strasbourg)
Alexei TSYGVINTSEV (École Normale Supérieure de Lyon)
Vladimir TURAEV (Université Louis Pasteur, Strasbourg)
Felix ULMER (Université de Rennes)
Hiroshi UMEMURA (Nagoya University)
Liane VALÈRE (Université de Savoie)
Alexander VARCHENKO (University of North Carolina, Chapel Hill)
Jean-François VIAUD (Université de La Rochelle)
Dan VOLOK (Delft University of Technology)
Ilya VYUGIN (Moscow State University)
Sergei YAKOVENKO (Weizmann Institute of Science, Rehovot)
Masaaki YOSHIDA (Kyushu University)